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COBORDISM OF DISK KNOTS

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ABSTRACT

We study cobordisms and cobordisms rel boundary of PL locally-flat disk knots $D^{n-2} \hookrightarrow D^n$. Any two disk knots are cobordant if the cobordisms are not required to fix the boundary sphere knots, and any two even-dimensional disk knots with isotopic boundary knots are cobordant rel boundary. However, the cobordism rel boundary theory of odd-dimensional disk knots is more subtle. Generalizing results of J. Levine on the cobordism of sphere knots, we define disk knot Seifert matrices and show that two higher-dimensional disk knots with isotopic boundaries are cobordant rel boundary if and only if their disk knot Seifert matrices are algebraically cobordant. We also ask which algebraic cobordism classes can be realized given a fixed boundary knot and provide a complete classification when the boundary knot has no 2-torsion in its middle-dimensional Alexander module.

In the course of this classification, we establish a close connection between the Blanchfield pairing of a disk knot and the Farber–Levine torsion pairing of its boundary knot (in fact, for disk knots satisfying certain connectivity assumptions, the disk knot Blanchfield pairing will determine the boundary Farber–Levine pairing). In addition, we study the dependence of disk knot Seifert matrices on choices of Seifert surface, demonstrating that all such Seifert matrices are **rationally** S-equivalent, but not necessarily integrally S-equivalent.

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1. Introduction

The cobordism theory of locally-flat knots was introduced by Fox and Milnor [5] in order to determine when a PL embedding of a 2-manifold in a 4-manifold can be made locally-flat by modifying the embedding only in neighborhoods of the isolated singularities. In any dimension, since neighborhood pairs of isolated singular points of codimension two manifold embeddings are isomorphic to cones on PL locally-flat **link knots** of spheres, such a replacement is possible if and only if the link knot embedding $K : S^{n-2} \hookrightarrow S^n$ can be extended to a locally-flat embedding $L : D^{n-1} \hookrightarrow D^{n+1}$; if this is possible, then K is called **null-cobordant** or **slice**. This definition quickly leads to a broader definition in which two locally-flat knots $K_0, K_1 : S^{n-2} \hookrightarrow S^n$ are deemed cobordant if and only if there exists a locally-flat proper embedding $\mathfrak{K} : S^{n-2} \times I \hookrightarrow S^n \times I$ such that $\mathfrak{K}|_{S^{n-2}\times 0} = K_0$ and $\mathfrak{K}|_{S^{n-2}\times 1} = -K_1$, which is K_1 with the reversed orientation. Then a knot is slice if and only if it is cobordant to the trivial knot.

It turns out that the set of cobordism equivalence classes of locally-flat sphere knots of a given dimension form a group, the operation being knot sum, and these groups were completely classified for $n \geq 4$ by Kervaire [9] and Levine

[13]: Kervaire showed that all even-dimensional knots are slice, while Levine demonstrated the equivalence of the odd-dimensional cobordism groups with algebraic cobordism groups of Seifert matrices. Later, Kearton [8] employed the work of Trotter [19] to demonstrate that this characterization is equivalent to an algebraic characterization in terms of certain cobordism groups of Blanchfield pairings on Alexander modules. By contrast, the study of cobordisms of classical knots $S^1 \hookrightarrow S^3$ remains an active field of research; see [16].

In general, however, singular points of codimension two piecewise-linear embeddings will not be isolated and so the link knots of points may not be locally-flat. Thus, the issue of simplifying local embeddings in more complex situations will necessarily involve a cobordism theory for non-locally-flat knots. It is tempting to declare all such knots null-cobordant by taking cones on them, since one is working in a category that does not require local-flatness. However, the goal most in keeping with the original Fox–Milnor treatment is to reduce the codimension of singularities; thus coning is not a satisfactory solution. Several more appropriate formulations for a cobordism theory of non-locally-flat sphere knots present themselves, but we will treat here only the next most general case after the classical one: cobordisms between knots with a single fixed singularity. This theory has a pleasant reformulation in terms of disk knots, and the results presented here are crucial to planned future work, in which we study cobordisms of sphere knots with arbitrary isolated singularities. We shall see that disk knots are also interesting in their own right, possessing in some sense the relationship to sphere knots that manifolds with boundary have to closed manifolds. For example, we shall see that the Blanchfield pairing of an odd-dimensional disk knot is related to the Farber–Levine pairing of its boundary knot in much the same way that the intersection pairing of an even-dimensional manifold is related to the linking pairing on its boundary.

Specifically, we define a **disk knot** to be a PL locally-flat proper embeddings $L : D^{n-2} \hookrightarrow D^n$. Since the embeddings are proper, taking boundary to boundary, each disk knot L determines a locally-flat sphere knot K on restriction to the boundary. We will call two disk knots L_0, L_1 cobordant if there exists a proper locally-flat PL embedding $\mathfrak{L} : D^{n-2} \times [0,1] \hookrightarrow D^n \times [0,1]$ such that $\mathfrak{L}|D^{n-2} \times 0 = L_0$ and $\mathfrak{L}|S^{n-2} \times 1 = -L_1$. Note that the restriction of $\mathfrak{L}$ to $\partial D^{n-2} \times [0,1]$ provides a cobordism between the boundary sphere knots K_0 and K_1. If this cobordism extends to an ambient isotopy of K_0 to K_1, we will call $\mathfrak{L}$ a cobordism rel boundary. This is the case that corresponds to cobordisms of sphere knots with fixed isolated singularities: gluing in I times the cone on K_0 gives a cobordism of sphere knots that fixes a neighborhood of the singularity. Conversely, given a cobordism of sphere knots with equivalent lone singularities and which is standard for a neighborhood of the singularity, we can remove this neighborhood to obtain a cobordism of disk knots rel boundary.

We should also note that since each disk knot is a **slicing disk** of its boundary knot, by studying cobordisms rel boundary of disk knots, we seek to classify precisely such slicing disks up to their own cobordisms. So in some sense we are studying a second order of cobordism theory. The results of this theory thus extend the original Fox–Milnor theory by providing some measure of the number of ways, up to cobordism, in which a codimension two embedding of a manifold with isolated singularities can be converted into a locally-flat embedding via the local replacement of singularities.

We now outline more precisely our main results.

It turns out that cobordisms of disk knots that do not fix the boundary as well as cobordisms rel boundary of even dimensional disk knots can be classified immediately via basic arguments:

Proposition 1.1 (Proposition 3.4): *If n is even, then any two disk knots $L_0, L_1 : D^{n-2} \hookrightarrow D^n$ are cobordant.*

Proposition 1.2 (Proposition 3.3): *If n is even, then any two disk knots $L_0, L_1 : D^{n-2} \hookrightarrow D^n$ with isotopic boundary knots are cobordant rel boundary.*

Proposition 1.3 (Proposition 3.5): *If n is odd, then any two disk knots $L_0, L_1 : D^{n-2} \hookrightarrow D^n$ are cobordant.*

This leaves the more challenging case of cobordism rel boundary for odd dimensional disk knots. To study this case, we will need to introduce Seifert matrices for disk knots. As opposed to Seifert matrices for sphere knots, which arise as certain forms on the middle dimensional homology of Seifert surfaces, Seifert matrices for disk knots are forms defined only on certain quotient homology modules (see Section 2, below, for the precise definition). Disk knot Seifert matrices also differ from those for sphere knots in that, if A is a disk knot Seifert matrix, the matrix $A + (-1)^n A'$ need not be integrally unimodular, only rationally so. Nonetheless, algebraic cobordism is well-defined on this larger class of matrices, and we attain the following conclusion:

THEOREM 1.4 (Theorem 3.6): *Let $L_0, L_1 : D^{2n-1} \hookrightarrow D^{2n+1}$, $n > 1$, be two disk knots with the same boundary knot. Let A_0 and A_1 be Seifert matrices for L_0 and L_1, respectively. Then L_0 and L_1 are cobordant rel boundary if and only if A_0 and A_1 are cobordant.*

Several interesting corollaries follow:

COROLLARY 1.5 (Corollary 4.1): *Suppose that L_0 and L_1 are disk knots $D^{2n-1} \hookrightarrow D^{2n+1}$, $n > 1$, such that $\partial L_0 = \partial L_1 = K$. Then a necessary condition for L_0 and L_1 to be cobordant rel boundary is that the product of the middle-dimensional Alexander polynomials $c_n^{L_0}(t)c_n^{L_1}(t)$ be similar in $\mathbb{Q}[t, t^{-1}]$ to a polynomial of the form $p(t)p(t^{-1})$.*

THEOREM 1.6 (Theorem 4.3): *Let L_0 and L_1 be two disk knots with common boundary K, $D^{2n-1} \hookrightarrow D^{2n+1}$, $n > 1$. Then there exists a sphere knot $\mathcal{K} : S^{2n-1} \subset S^{2n+1}$ such that L_0 is cobordant to the knot sum (away from the boundary) $L_1 \# \mathcal{K}$.*

THEOREM 1.7 (Theorem 4.5): *Given any disk knot $L : D^{2n-1} \hookrightarrow D^{2n+1}$, $n > 1$, L is cobordant rel boundary to a disk knot L_1 such that $\pi_i(D^{2n+1} - L_1) \cong \pi_i(S^1)$ for $i < n$.*

This last theorem tells us that every disk knot is cobordant rel boundary to a **simple disk knot**.

The next question to consider is that of which cobordism classes of matrices arise as the Seifert matrices of disk knots. We will show that all such possible matrices occur for knots of sufficiently high dimension, but we will also be interested in the sharper question of which classes arise for disk knots given a fixed boundary knot. At this point it will be useful to invoke the technology of Blanchfield pairings and Farber–Levine torsion pairings of Alexander modules. It is a theorem of Trotter [19] that, for sphere knots, Seifert matrices determine isometric Blanchfield pairings if and only if they are S-equivalent, and hence it is possible to restate the cobordism results concerning sphere knots in terms of properties of these Blanchfield pairings (see Kearton [8]). The Farber–Levine torsion pairing [15, 2] is less known, though it has been used in certain classification schemes of simple knots [10, 3, 4]. We will establish a relation between these two pairings. In fact, if a disk knot is simple, i.e. its complement has the homotopy groups of a circle below the “middle” dimension, its Blanchfield pairing will completely determine the Farber–Levine pairing of its boundary knot. In particular, letting $\tilde{C}$ denote the infinite cyclic cover of the disk knot complement and $\tilde{X}$ the infinite cyclic cover of the complement of its boundary sphere knot, we can prove the following:

THEOREM 1.8 (Theorem 6.1): *Given a simple disk knot $L : D^{2n-1} \hookrightarrow D^{2n+1}$, the $\mathbb{Z}[\mathbb{Z}]$-module $H_{n-1}(\tilde{X})$ and the Farber-Levine $\mathbb{Z}$-torsion pairing on its $\mathbb{Z}$-torsion submodule $T_{n-1}(\tilde{X})$ are determined up to isometry by the isometry class of the Blanchfield self-pairing on $H_n(\tilde{C})$.*

THEOREM 1.9 (Theorem 6.2): *For a simple disk knot $L : D^{2n-1} \hookrightarrow D^{2n+1}$, the $\mathbb{Z}[\mathbb{Z}]$-module $T_{n-1}(\tilde{X})$ and its Farber-Levine $\mathbb{Z}$-torsion pairing are determined up to isometry by the isometry class of $\operatorname{cok}(H_n(\tilde{X}) \to H_n(\tilde{C}))$ with its Blanchfield self-pairing.*

COROLLARY 1.10 (Corollary 6.3): *For a simple disk knot $L : D^{2n-1} \hookrightarrow D^{2n+1}$, the $\mathbb{Z}[\mathbb{Z}]$-module $T_{n-1}(\tilde{X})$ and its Farber–Levine $\mathbb{Z}$-torsion pairing are determined up to isometry by any Seifert matrix for L.*

These theorems, together with a theorem of Kojima [10], will allow us to prove that, given a fixed boundary knot K of sufficiently high dimension and with no middle-dimensional 2-torsion, any cobordism class of matrices containing an element that correctly determines the Farber–Levine pairing of K is realizable as the matrix cobordism class of a disk knot with K as its boundary knot. See Theorem 5.10 for a more accurate statement.

In the course of these investigations, we will also need to engage in an in-depth study of how the Seifert matrix of a disk knot varies with choice of Seifert surface. In particular, we will prove the following theorem:

THEOREM 1.11 (Theorem 7.1): *Any two Seifert matrices for a disk knot differ by a rational S-equivalence.*

The organization of this paper is as follows: In Section 2, we present the basic definitions and technical details concerning Seifert matrices of disk knots. In Section 3, we begin our investigation and determine when two disk knots are cobordant. In Section 4, we present the aforementioned corollaries of this algebraic classification. Section 5 contains the constructions that allow us to realize the algebraic cobordism matrices geometrically. Sections 6 is dedicated to the relation between disk knot Blanchfield pairings and their boundary sphere knot Farber–Levine pairings. Finally, Section 7 contains the calculations of how disk knot Seifert matrices change as the Seifert surface is varied.

2. Preliminaries

In this section, we introduce our basic definitions, background material, and notation.

2.1. Knots, complements, Seifert surfaces, and Alexander modules. A piecewise linear (PL) locally-flat **disk knot** is an embedding $L : D^{n-2} \hookrightarrow D^n$, where D^n is the standard PL disk of dimension n. All disk knots will be proper embeddings, i.e., $L(D^{n-2}) \cap \partial D^n = L(\partial D^{n-2})$, and there is a collar of the boundary in which the embedding is PL-homeomorphic to $(\partial D^n, L(\partial D^{n-2})) \times I$. The locally-flat condition means that for each point $x \in L(D^{n-2})$, there is a neighborhood U such that $(U, U \cap L(D^{n-2})) \cong_{PL} (B^n, B^{n-2})$, the standard (closed) ball pair. The boundary embedding $\partial D^{n-2} \hookrightarrow \partial D^n$ is the PL locally-flat **boundary sphere knot** $K : S^{n-3} \hookrightarrow S^{n-1}$. Often, we will employ the standard abuse of notation and confuse the symbols for the maps L and K with those for their images.

We let C denote the **exterior** of L, the complement of an open regular neighborhood of L; C is homotopy equivalent to $D^n - L$. We let X denote $C \cap \partial D^n$, the exterior of K. Using Alexander duality (respectively, Alexander duality for a ball; see [17, p. 426]), X and C are homology circles and so possess **infinite cyclic covers** that we denote $\tilde{X}$ and $\tilde{C}$. F denotes a Seifert surface for K, and V denotes a **Seifert surface** for L, i.e., an oriented bi-collared $n-1$ dimensional submanifold of D^n whose boundary is the union of L and a Seifert surface for the boundary knot K. Such Seifert surfaces always exist (see [7]). Note that $H_*(\partial V) \cong H_*(F)$ for $* \leq n-3$.

The groups $H_*(\tilde{X})$, $H_*(\tilde{C})$, and $H_*(\tilde{C}, \tilde{X})$, which we call the **Alexander modules**, inherit structures as modules over the ring of Laurent polynomials $\Lambda = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$ by the action of the covering translation. A Λ-module is of **type** K if it is finitely generated and multiplication by $t-1$ acts as an automorphism. Equivalently, a Λ-module of type K is a finitely generated $\Lambda[(t-1)^{-1}]$ module. It is well-known that $H_*(\tilde{X})$ is a torsion Λ-module of type K for $* > 0$ (see e.g., [15]). Since C is a homology circle, $H_*(\tilde{C})$ is also of type K for $* > 0$ by Levine [15, Proposition 1.2] since the proof of this proposition only relies on

C being a homology circle. It then follows from [15, Corollary 1.3] that $H_*(\tilde{C})$, $* > 0$, is a Λ-torsion module. Hence so is $H_*(\tilde{C}, \tilde{X})$ from the reduced long exact sequence of the pair (in fact, it is similarly of type K by the five lemma applied to the long exact sequence of the pair under multiplication by $t-1$).

A sphere knot $S^{n-2} \hookrightarrow S^n$ is called **simple** if $\pi_i(X) \cong \pi_i(S^1)$ for $i \leq (n-2)/2$. By [12] this is as connected as the complement of a locally-flat knot can be without the knot being trivial. We similarly define a disk knot $D^{n-2} \hookrightarrow D^n$ to be **simple** if $\pi_i(C) \cong \pi_i(S^1)$ for $i \leq (n-2)/2$.

Finally, we recall that a pairing of modules $(\, , \,) : A \otimes B \to C$ is called nondegenerate if $(a, b) = 0$ for all $b \in B$ implies $a = 0$ and if $(a, b) = 0$ for all $a \in A$ implies $b = 0$. We call the pairing nonsingular, if $a \to (a, \cdot)$ is an isomorphism $A \to \operatorname{Hom}(B, C)$ and $b \to (\cdot, b)$ is an isomorphism $B \to \operatorname{Hom}(A, C)$. A rational matrix is nondegenerate and nonsingular if its determinant is not 0. An integer matrix is considered nondegenerate if its determinant is nonzero and nonsingular if its determinant is ± 1. The pairings we shall be most concerned with are the Blanchfield and Farber–Levine pairings on Alexander modules. Detailed descriptions of these pairings are contained below in Section 6.

2.2. The Seifert matrix. The main algebraic invariant in our study of cobordism will be the Seifert matrix of a disk knot. Recall that for an odd-dimensional sphere knot $K : S^{2n-1} \to S^{2n+1}$ with Seifert surface F^{2n}, the Seifert matrix is traditionally defined as follows: Let $F_n(F) = H_n(F)/\text{torsion}$, let $i_- : F_n(F) \to F_n(S^{2n+1} - F)$ be induced by displacing F along its bicollar in the negative direction, and let $L : F_n(S^{2n+1} - F) \otimes F_n(F) \to \mathbb{Z}$ be the Alexander linking pairing. Then a **Seifert matrix** is the matrix (with respect to some chosen basis) of the pairing $F_n(F) \otimes F_n(F) \to \mathbb{Z}$ given by $x \otimes y \to L(i_-(x), y)$,

Suppose instead we start with a disk knot $L : D^{2n-1} \hookrightarrow D^{2n+1}$ with Seifert surface V and boundary Seifert surface F of the boundary sphere knot. Now we must work with relative pairings. We first observe that $H_i(V, F) \cong H_i(V, \partial V)$ for $i \leq 2n-2$ and $H_{2n-1}(V, F) \to H_{2n-1}(V, \partial V)$ is onto. So, in particular, $H_n(V, F) \cong H_n(V, \partial V)$, induced by inclusion, for $n \geq 2$.

The appropriate analogue of the Seifert matrix is defined on the cokernel of the homomorphism induced by inclusion $H_n(F) \to H_n(V)$, mod torsion. We denote $\bar{E} = \operatorname{cok}(H_n(F) \to H_n(V))/\text{torsion}$, and we define the **disk knot Seifert matrix** to be the matrix θ of the pairing (with respect to some fixed chosen basis) $\bar{E} \otimes \bar{E} \to \mathbb{Z}$ given by $x \otimes y \to L''(i_- p_*(x), y)$, where p_* is induced by the natural projection

$$F_n(V) \to F_n(V, F),$$

$i_- : F_n(V,F) \to F_n(D^{2n+1} - V, S^n - F)$ is again induced by pushing along the bicollar of the Seifert surface, and $L'' : F_n(D^{2n+1} - V, S^n - F) \otimes F_n(V) \to \mathbb{Z}$ is the linking pairing induced by Alexander duality on a ball (see [17]). Note that this is well-defined for $y \in \bar{E}$ since this pairing is trivial on elements in $\mathrm{Im}(H_n(F) \to H_n(V))$.

It is not difficult to check, using the properties of the various pairings involved, that if we let θ' denote the transpose of θ, then $-\theta - (-1)^n\theta'$ is the nondegenerate intersection pairing $T : \bar{E} \otimes \bar{E} \to \mathbb{Z}$ given by $T(x \otimes y) = S(p_*x, y)$, where p_* is as above and $S : F_n(V,F) \otimes F_n(V) \to \mathbb{Z}$ is the nonsingular Lefschetz–Poincaré duality pairing. Once again, T is well-defined on $\bar{E}$ since $S(x,y) = 0$ if $y \in \mathrm{Im}(H_n(F) \to H_n(V))$. In particular, we see that $\det(\theta + (-1)^n\theta') \neq 0$.

We note that there is a correspondence between PL locally-flat sphere knots and PL-locally-flat disk knots whose boundary knots are trivial: Given such a disk knot, we can cone the boundary to obtain a **locally-flat** sphere knot, and conversely, given a sphere knot, we can remove a ball neighborhood of any point on the knot to obtain a disk knot with trivial boundary. If we then consider a Seifert surface for such a disk knot whose boundary Seifert surface is the trivial disk Seifert surface for the boundary unknot, then $H_n(F) = 0$, and the disk knot Seifert matrix θ will be the classical sphere knot Seifert matrix for the corresponding sphere knot.

It will be useful for what follows to compare the disk knot Seifert matrix θ just defined, with certain matrices arising in a similar context in [7]. In [7], we studied presentations of Alexander modules in terms of matrices denoted R, τ, and μ. The matrix R represented an isomorphism from what we have here called $\bar{E}$ to

$$\ker(\partial_* : H_n(V,F) \to H_{n-1}(F))/\text{torsion},$$

where the latter group is given a basis dual to that of $\bar{E}$ by Lefschetz duality. Equivalently, R is the transpose of the matrix of the pairing T (with respect to the same basis by which we obtain θ). τ and μ are the matrices of the respective homomorphisms i_{-*} and i_+* from $\ker(\partial_* : H_n(V,F) \to H_{n-1}(F))/\text{torsion}$ to $\ker(\partial_* : H_n(D^{2n+1} - V, S^{2n} - F) \to H_{n-1}(S^{2n} - F))/\text{torsion}$, induced by pushing V into its bicollar. The matrices are with respect to the given fixed basis of $\bar{E}$

and the dual bases induced on the other groups via Lefschetz and Alexander duality. More details can be found in [7].

It is not hard to show from the definitions of these matrices and by applying pairings that

$$\theta = (\tau R)' = (-1)^n \mu R.$$

It is shown in [7] that $(-1)^{q+1}(R^{-1})'\tau R t - \tau'$, is a presentation matrix for the middle dimensional Alexander module $\operatorname{cok}(H_n(\tilde{C};\mathbb{Q}) \to H_n(\tilde{C},\tilde{X};\mathbb{Q}))$ as a module over $\Gamma = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[\mathbb{Z}]$, while the matrix $\frac{1-t}{(R^{-1})'\tau t-(-1)^{q+1}\tau' R^{-1}}$ represents the Blanchfield pairing of this module (see Section 6, below, for definitions). Both of these matrices are with respect to natural **integral** bases within the rational modules.

3. Disk knot cobordism

Let $L : D^{n-2} \hookrightarrow D^n$ be a disk knot. We define two types of cobordism between disk knots:

Definition 3.1: Two disk knots L_0, L_1 are cobordant if there exists a proper embedding $F : D^{n-2} \times I \hookrightarrow D^n \times I$ such that $F|D^{n-2} \times i = L_i \times i$ for $i = 0, 1$ and $F|\partial D^{n-2} \times I$ is a cobordism of the boundary sphere knots K_0, K_1.

Definition 3.2: Two disk knots L_0, L_1 are cobordant rel boundary if the boundary knots of L_0 and L_1 are ambient isotopic and there exists a cobordism from L_0 to L_1 that restricts to this isotopy on $\partial D^{n-2} \times I$.

N.B. Due to the usual orientation switch of the total space from the bottom to the top of a cylinder, the embedding $L_1 \times 1$ actually represents the knot $-L_1$, the mirror image of L_1. This, in particular, will be the case when we consider $L_1 \times 1 = -L_1$ as a submanifold of $S^n = \partial(D^n \times I)$. The orientation of the embedded knot is itself switched, but this orientation usually plays no role in higher-dimensional knot theory so we omit further mention.

We first treat cobordisms of even-dimensional disk knots, which can be dispensed with rather quickly, and then move on to odd-dimensional disk knots, whose theory, at least for cobordisms rel boundary, is much more complicated.

3.1. Cobordism of even-dimensional disk knots.

Proposition 3.3: *If n is even, then any two disk knots $L_0, L_1 : D^{n-2} \to D^n$ with isotopic boundary knots are cobordant rel boundary.*

Proof. Given two such knots L_0, L_1, the maps $L_0 \times i$ on $D_{n-2} \times i$ along with the isotopy H connecting their boundary knots determines a sphere knot $S^{n-2} \to S^n = \partial(D^n \times I)$ by

$$(L_0 \times 0) \cup H \cup (L_1 \times 1) : \; S^{n-2} = (D^{n-2} \times 0) \cup (S^{n-3}) \times I \cup (D^{n-2} \times 1) \\ \to S^n = (D^n \times 0) \cup (S^{n-1}) \times I \cup (D^n \times 1).$$

This is an even dimensional sphere knot and so it is null-cobordant by Kervaire [9]. Any such null-cobordism provides the desired cobordism of the disk knots. ∎

Proposition 3.4: *If n is even, then any two disk knots $L_0, L_1 : D^{n-2} \to D^n$ are cobordant.*

Proof. The proof of the existence of a cobordism if the boundary knots are cobordant is the same as in the last proposition but connecting the boundary knots is done by their cobordism instead of the trace of an isotopy. However, any boundary knot of a disk knot is null-cobordant, so, in particular, the boundaries are cobordant to each other. ∎

3.2. Cobordism of odd-dimensional disk knots.

For odd-dimensional disk knots, all boundary sphere knots are cobordant by [9] since they are all even-dimensional.

Proposition 3.5: *If n is odd, then any two disk knots $L_0, L_1 : D^{n-2} \hookrightarrow D^n$ are cobordant.*

Proof. Let K_0, K_1 be even dimensional boundary knots of L_0 and L_1. As noted, all even dimensional knots are nullcobordant by Kervaire [9]. Let us construct the cobordism G of the boundary knots K_0 and K_1 as follows: Let $G|\partial D^{n-2} \times [0, 1/4]$ realize a cobordism of K_0 to the unknot. The union of L_0 with this nullcobordism gives a disk knot in $(D^n \times 0) \cup (\partial D^n \times [0, 1/4])$ with unknotted boundary knot. Let J_0 denote the sphere knot obtained by filling in this unknotted boundary; we can think of obtaining J_0 by taking the cone pair on the boundary of the disk knot (which will be a locally-flat sphere knot since the boundary knot is trivial). Define $-J_1$ similarly by adjoining a null-cobordism on $\partial D^n \times [3/4, 1]$ (recall that the embedding $L_1 \times 1$ represents the disk knot $-L_1$, taking into account orientations on the cylinder as induced from the 0 end). Now consider the knot $-(J_0\# - J_1) = (-J_0)\#J_1$, where $\#$ represents knot sum. By removing neighborhoods of two points on the knot, we can think of this knot as a cobordism between two trivial knots $S^{n-3} \subset S^{n-1}$, and we can glue this cobordism into $\partial D^n \times [1/4, 3/4]$, matching the ends since all unknots are ambient isotopic. So now we have constructed a cobordism from K_0 to K_1, and the knotted sphere in the boundary of $D^n \times I$ given by the union of L_0, $L_1 \times 1 = -L_1$, and the cobordism is $J_0\#((-J_0)\#J_1)\#(-J_1) = (J_0\#J_1)\# - (J_0\#J_1)$, which is null-cobordant. Again any null-cobordism now realizes the cobordism of disk knots. ∎

So there remains only the much more difficult consideration of cobordism of odd dimensional disk knots rel boundary. As seen in the preceding propositions and described in more detail below, the problem reduces to finding a null-cobordism of sphere knots composed of the union of L_0 and L_1. By [13] the cobordism class of a sphere knot $S^{2n-1} \hookrightarrow S^{2n+1}$, $n > 1$, is determined by its Seifert matrix. So we are left with the problem of determining Seifert matrices for disk knots joined along their boundaries. Note that if $n = 1$, the disk knot $D^1 \hookrightarrow D^3$ has trivial boundary and so the problem of determining cobordisms rel boundary is in this case equivalent to the problem of classifying cobordisms of classical knots, which remains an unsolved problem. Hence we concentrate on the cases $n > 1$, in which the disk knot and sphere knot theories are truly different (though closely related).

Following Levine in [13], a $2r \times 2r$ integer matrix is called **null-cobordant** if it is integrally congruent to a matrix of the form $\begin{pmatrix} 0 & N_1 \\ N_2 & N_3 \end{pmatrix}$, where each matrix N_i is $r \times r$. Two square integer matrices A and B are **cobordant** if $A \boxplus -B$ is null-cobordant, where $\boxplus$ denotes the block sum of matrices $A \boxplus -B = \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix}$. In [13], Levine shows that odd-dimensional sphere knots, $n > 1$ are cobordant if and only if their Seifert matrices are cobordant.

Our next goal is to prove the following theorem, which will take up the rest of this section.

Theorem 3.6: *Let A_0 and A_1 be Seifert matrices for disk knots*

$$L_0, L_1 : D^{2n-1} \hookrightarrow D^{2n+1}, \quad n > 1,$$

with the same boundary knot. L_0 and L_1 are cobordant rel boundary if and only if A_0 and A_1 are cobordant.

It is useful to reduce questions about cobordism of matrices to questions about **rational cobordism** - we will call a rational $2r \times 2r$ matrix A **rationally null-cobordant** if it is rationally congruent to a matrix of the form $\begin{pmatrix} 0 & N_1 \\ N_2 & N_3 \end{pmatrix}$. This is equivalent to saying that A is null-cobordant as a pairing of rational vector spaces $\mathbb{Q}^{2r} \times \mathbb{Q}^{2r} \to \mathbb{Q}$ given by $x \times y \to x'Ay$, i.e. there exists an r-dimensional subspace of $\mathbb{Q}^{2r}$ on which the restriction of the pairing is 0.

Lemma 3.7: *Let A be a $2r \times 2r$ integer matrix. Then A is null-cobordant if and only if it is rationally null-cobordant.*

Proof. If A is a $2r \times 2r$ integral null-cobordant matrix, then there is a rank r direct summand F of $\mathbb{Z}^{2r}$ on which A restricts to the 0 bilinear form. Hence this matrix is also rationally null-cobordant, restricting to the 0 form on $F \otimes \mathbb{Q}$.

Conversely, suppose that A is rationally null-cobordant so that there is an r-dimensional $\mathbb{Q}$ subspace V of $\mathbb{Q}^{2r} \cong \mathbb{Z}^{2r} \otimes \mathbb{Q}$ on which A restricts to the 0 bilinear form. Let L be the lattice $\mathbb{Z}^{2r} \cap V$. This is a free abelian subgroup of $\mathbb{Z}^{2r}$, in fact a direct summand, since any element of $\mathbb{Z}^{2r}$ that has a scalar multiple in L must also be in L. L must have rank at least r, since given r linear independent rational vectors in V, there are integral multiples of these vectors that lie in L (by clearing denominators of the coordinates), and these scalar multiples remain linearly independent over $\mathbb{Q}$ and hence over $\mathbb{Z}$. So A is the 0 form on a free abelian group of rank $\geq r$ that is a direct summand of $\mathbb{Z}^{2r}$. ∎

We say that two square rational matrices A and B are **rationally cobordant** if $A \boxplus -B$ is rationally null-cobordant, where $\boxplus$ denotes the block sum of matrices $A \boxplus -B = \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix}$.

Corollary 3.8: *Two* **integral matrices** *are integrally cobordant if and only if they are rationally cobordant.*

Proof. This is an immediate consequence of Lemma 3.7. ∎

N.B. Even though we will be concerned with rational cobordism classes, the term **Seifert matrix** will always refer to the **integral** Seifert matrix defined in Section 2 **unless explicitly stated otherwise**.

LEMMA 3.9: *Suppose A and N are square matrices of rational numbers, that N and $A \boxplus N$ are rationally null-cobordant, and that some rational linear combination $\lambda N + \mu N'$ has non zero determinant. Then A is rationally null-cobordant.*

Proof. The proof is the same as that of Levine's for integral null-cobordism [13, Lemma 1] replacing $\mathbb{Z}$ with $\mathbb{Q}$ in all steps. ∎

For Seifert matrices of sphere knots

$$S^{2n-1} \hookrightarrow S^{2n+1}$$

or disk knots

$$D^{2n-1} \hookrightarrow D^{2n+1},$$

these conditions will be satisfied with $\lambda = 1$, $\mu = (-1)^n$. For sphere knots, this is well-known (see [12] or [13]). For disk knots, this can be seen from the fact that the Alexander polynomials of disk knots are nonzero when evaluated at 1 (see [6] or [7] for details).

COROLLARY 3.10: *For fixed rational λ and μ, the set of rational cobordism classes of square rational matrices A satisfying $\det(\lambda A + \mu A') \neq 0$ is an abelian group under block sum, the inverse of the class represented by a matrix A being the class represented by $-A$.*

Proof. Again, this corollary follows from the lemma as in [13, §3] by replacing integral statements with rational ones. ∎

PROPOSITION 3.11: *Let A, $\hat{A}$ be Seifert matrices for the disk knot L. Then A and $\hat{A}$ are integrally cobordant.*

Proof. Let V and $\hat{V}$ be Seifert surfaces with respect to which A and $\hat{A}$ are the integral Seifert matrices. Then it follows from the results of Section 7, below, that A and B are related by a sequence of rational congruences and enlargements or reductions of the form

$$M \leftrightarrow M' = \begin{pmatrix} M & 0 & \eta \\ 0 & 0 & x \\ \xi & x' & y \end{pmatrix},$$

where M is a matrix, η is a column vector, ξ is a row vector, x, x', and y are integers, and all "0"s represent the necessary 0 entries to make this matrix square. Also, either x or x' is 0 while the other is nonzero. So it suffices to show that $-M \boxplus M'$ is rationally nullcobordant. If M is a $k \times k$ matrix, let I_j be the $j \times j$ identity matrix, and let $P = \begin{pmatrix} I_k & I_k & 0 \\ 0 & I_k & 0 \\ 0 & 0 & I_2 \end{pmatrix}$. Then

$$P'(-M \boxplus M')P = \begin{pmatrix} -M & -M & 0 & 0 \\ -M & 0 & 0 & \eta \\ 0 & 0 & 0 & x \\ 0 & \xi & x' & y \end{pmatrix}$$

contains a $k+1 \times k+1$ dimensional 0 matrix block symmetric with respect to the diagonal, so it is rationally null-cobordant. The integral cobordism is then implied by Corollary 3.8. ∎

If we have two disk knots $L_0, L_1 : D^{2n-1} \hookrightarrow D^{2n+1}$ with the same boundary sphere knot, then we can form a sphere knot $L_0 \cup_\partial -L_1 : S^{2n-1} \hookrightarrow S^{2n+1}$ by gluing the two knots together along their common boundary, after appropriately reversing the orientation of L_1. Theorem 3.6 will now follow from the following theorem.

Theorem 3.12: *Let A_0 and A_1 be Seifert matrices for disk knots*

$$L_0, L_1 : D^{2n-1} \hookrightarrow D^{2n+1}, \quad n > 1,$$

with the same boundary knot. Then the Seifert matrix of the sphere knot $L_0 \cup_\partial -L_1$ is integrally cobordant to $A_0 \boxplus -A_1$.

Let us first see that Theorem 3.12 implies Theorem 3.6.

Proof of Theorem 3.6. We can think of $L_0 \cup_\partial -L_1$ as embedding $S^{2n-1} = D^{2n-1} \cup (S^{2n-2} \times I) \cup D^{2n-1}$ into $\partial(D^{2n+1} \times I)$ as the union of L_0 on $D^{2n-1} \times 0$, L_1 on $D^{2n-1} \times 1$, and an isotopy of $\partial L = K$ on $S^{2n-2} \times I$. Thus finding a disk knot cobordism rel boundary from L_0 to L_1 is equivalent to showing that $L_0 \cup_\partial -L_1$ is a null-cobordant sphere knot. By [13], a sphere knot is null-cobordant if and only if its Seifert matrix is null cobordant.

If the matrices A_0 and A_1 for L_0 and L_1 are cobordant, then, by Theorem 3.12, the integral Seifert matrix for $L_0 \cup_\partial -L_1$, which is cobordant to $A_0 \boxplus -A_1$, is nullcobordant. Thus $L_0 \cup_\partial -L_1$ is nullcobordant. Conversely, if A_0 and A_1 are not cobordant, then $A_0 \boxplus -A_1$ is not integrally nullcobordant and so $L_0 \cup_\partial -L_1$ is not null-cobordant. ∎

Now we must work toward Theorem 3.12.

Let L_0 and L_1 be two $2n+1$ disk knots with the same boundary knot K, $n > 1$, and let $\mathfrak{K} = L_0 \cup_K -L_1$. Let V_0 and V_1 be Seifert surfaces for L_0 and L_1 with boundary Seifert surfaces F_0 and F_1 for K (see [7]). Then there is a cobordism Υ of Seifert surfaces from F_0 to F_1 with boundary the union of F_0, $-F_1$, and the trace of an isotopy of K by [14, §3]. We can form a Seifert surface for $\mathfrak{K}$ by $W = V_0 \cup_{F_0} \Upsilon \cup_{-F_1} -V_1$. Since the union of L_0 with the trace of an isotopy of its boundary is isotopic to L_0, we will simplify notation by combining V_0 and Υ to form a new V_0. So we can consider W to be composed of Seifert surface V_0 and $-V_1$ for L_0 and $-L_1$, joined along a single Seifert surface F for K.

In what follows, we use the isomorphism of the groups $H_*(V_1) \cong H_*(-V_1)$ to simplify notation.

We consider the Mayer–Vietoris sequence

$$\longrightarrow H_n(F) \xrightarrow{j} H_n(V_0) \oplus H_n(V_1) \xrightarrow{\rho} H_n(W) \xrightarrow{\partial} H_{n-1}(F) \xrightarrow{j'} .$$

We are first interested in computing ranks of free abelian subgroups, so we can consider homology groups with **rational** coefficients (though we omit them from the notation for clarity). Then there is a splitting $H_n(W) \cong \operatorname{im}(\partial) \oplus \operatorname{cok}(j)$.

Now from the **rational** long exact sequences of the pairs (V_s, F), $s = 0, 1$:

$$\text{(1)} \qquad \longrightarrow H_n(F) \xrightarrow{i_s} H_n(V_s) \xrightarrow{p_s} H_n(V_s, F) \xrightarrow{\partial_s} H_{n-1}(F) \xrightarrow{i'_s},$$

$H_n(V_s) \cong \operatorname{cok}(i_s) \oplus \operatorname{im}(i_s)$, and, furthermore, $\operatorname{im}(j) \subset \operatorname{im}(i_0) \oplus \operatorname{im}(i_1)$, so we can write $\operatorname{cok}(j) \cong \frac{\operatorname{cok}(i_0) \oplus \operatorname{cok}(i_1) \oplus \operatorname{im}(i_0) \oplus \operatorname{im}(i_1)}{\operatorname{im}(j)} \cong \operatorname{cok}(i_0) \oplus \operatorname{cok}(i_1) \oplus \frac{\operatorname{im}(i_0) \oplus \operatorname{im}(i_1)}{\operatorname{im}(j)}$.

Now $\operatorname{cok}(i_s)$ is the group on which the Seifert matrix of L_s is defined. We need to study the other summands $(\operatorname{im}(i_0) \oplus \operatorname{im}(i_1))/\operatorname{im}(j)$ and $\operatorname{im}(\partial)$ of $H_n(W)$. We claim that these two summands have the same dimension.

Let $|G|$ stand for the dimension of the vector space G. Suppose that $|H_n(F)| = m$ and $|H_n(V_s)| = M_s$. Then $|\operatorname{im}(i_s)| = |\operatorname{coim}(i_s)| = m - |\ker(i_s)|$, and $|\operatorname{im}(j)| = m - |\ker(j)| = m - |\ker(i_0) \cap \ker(i_1)|$. So,

$$\begin{aligned} \left| \frac{\operatorname{im}(i_0) \oplus \operatorname{im}(i_1)}{\operatorname{im}(j)} \right| &= m - |\ker(i_0)| + m - |\ker(i_1)| - (m - |\ker(i_0) \cap \ker(i_1)|) \\ &= m + |\ker(i_0) \cap \ker(i_1)| - |\ker(i_0)| - |\ker(i_1)|. \end{aligned}$$

Now, since F is a $2n-1$ manifold with sphere boundary, and since V_s is a $2n$-manifold whose boundary is the union of F with a disk, Poincaré duality holds, and, in particular, $|H_n(F)| = |H_{n-1}(F)| = m$ and $|H_n(V_s)| =$

$|H_n(V_s, F)| = M_s$. Let us fix a basis of $H_{n-1}(F)$ and use the standard orthonormal inner product with respect to this basis to identify $H_{n-1}(F)$ with $\mathrm{Hom}(H_{n-1}(F); \mathbb{Q}) \cong H^{n-1}(F; \mathbb{Q}) \cong H_n(F; \mathbb{Q})$. Consider now $\ker(i_s)$. Under this identification, via Poincaré duality, $\ker(i_s) = (\ker(i'_s))^{\perp}$. Indeed, if $x \in \ker(i_s)$ and $y \in \ker(i'_s) = \mathrm{im}(\partial_s)$ so $y = \partial_s z$, then on the intersection pairing, we have $S_F(x, y) = S_V(i_s(x), z) = 0$; so the identification takes $\ker(i_s)$ into $(\ker(i'_s))^{\perp}$. But we also have $|\ker(i_s)| = m - |\mathrm{im}(i_s)| = m - |\ker(p_s)| = m - (M_s - |\mathrm{im}(p_s)|) = m - (M_s - |\ker(\partial_s)|) = m - (M_s - (M_s - |\mathrm{im}(\partial_s)|)) = m - |\ker(i'_s)|$, so $\ker(i_s) = (\ker(i'_s))^{\perp}$. Then we compute

$$\begin{aligned}\left|\frac{\mathrm{im}(i_0) \oplus \mathrm{im}(i_0)}{\mathrm{im}(j)}\right| &= m + |\ker(i_0) \cap \ker(i_1)| - |\ker(i_0)| - |\ker(i_1)| \\ &= m + |(\ker(i'_0))^{\perp} \cap (\ker(i'_1))^{\perp}| - |(\ker(i'_0))^{\perp}| - |(\ker(i'_1))^{\perp}| \\ &= m - |(\ker(i'_0))^{\perp} + (\ker(i'_1))^{\perp}| \\ &= |\ker(i'_0) \cap \ker(i'_1)| \\ &= |\ker j'| \\ &= |\mathrm{im}(\partial)|.\end{aligned}$$

Here, the fourth equality uses that, in a vector space X with subspaces Y and Z, $(Y^{\perp} + Z^{\perp})^{\perp} = Y \cap Z$.

So once again, with rational coefficients, we can write

$$H_n(W) \cong \mathrm{cok}(i_0) \oplus \mathrm{cok}(i_1) \oplus \frac{\mathrm{im}(i_0) \oplus \mathrm{im}(i_1)}{\mathrm{im}(j)} \oplus \mathrm{im}(\partial),$$

where the last two summands have the same dimension. Let us denote $U = \frac{\mathrm{im}(i_0) \oplus \mathrm{im}(i_1)}{\mathrm{im}(j)}$

We next observe that the Seifert form is 0 when restricted to

$$U \times (\mathrm{cok}(i_0) \oplus \mathrm{cok}(i_1)) \text{ or } (\mathrm{cok}(i_0) \oplus \mathrm{cok}(i_1)) \times U.$$

This is true because any element of $\mathrm{cok}(i_s)$ can be represented by a cycle lying entirely in the interior of V_s and hence of $D^{2n+1} \times s$ in the cobordism, and the same is true for any translate along a normal vector to the Seifert surface. Also, we can then find a chain in $D^{2n+1} \times s$ whose boundary is the push in the bicollar of our cycle. Meanwhile, any element of U can be represented by a cycle that lies in $\partial D^{2n+1} \times I$, and the same for its translates along the bicollar, and a choice of chain it bounds in $\partial D^{2n+1} \times I$. So then clearly the linking numbers of any such cycles must be 0.

At last we can prove theorem 3.12.

Proof of Theorem 3.12. Let V_0, V_1, F, and W be as above. Let B_0 and B_1 be the Seifert matrices of L_0 and L_1 corresponding to these Seifert surfaces. Then by Proposition 3.11, B_0 and B_1 are rationally cobordant to A_0 and A_1, respectively.

Now we consider the Seifert matrix M determined by W and show that it is rationally cobordant to $A_0 \boxplus -A_1$, which will suffice to prove the theorem.

We know that $H_n(W;\mathbb{Q}) \cong \text{cok}(i_0) \oplus \text{cok}(i_1) \oplus \frac{\text{im}(i_0)\oplus\text{im}(i_1)}{\text{im}(j)} \oplus \text{im}(\partial)$, and the Seifert pairings on $\text{cok}(i_0)$ and $\text{cok}(i_1)$ must be restricted to B_0 and $-B_1$ by definition (the negative is due to the reverse of orientation by considering L_1 in $D^{2n+1} \times 1 \subset D^{2n+1} \times I$). Furthermore, these subspaces are orthogonal under the Seifert pairing since elements of $\text{cok}(i_0)$ are represented by chains in $V_0 \subset D^{2n+1} \times 0 \subset \partial(D^{2n+1} \times I)$, while elements of $\text{cok}(i_1)$ are represented by chains in $V_1 \subset D^{2n+1} \times 1 \subset \partial(D^{2n+1} \times I)$, so these chains cannot link in $\partial(D^{2n+1} \times I) = S^{2n+1}$. Similarly, elements in $\frac{\text{im}(i_0)\oplus\text{im}(i_0)}{\text{im}(j)}$ can be represented by chains in F, that can be pushed into either V_0 or V_1, and so they do not link with each other or elements of $\text{cok}(i_0)$ and $\text{cok}(i_1)$. Thus M must have the form (up to rational change of basis and hence rational cobordism)

$$M = \begin{pmatrix} B_0 & 0 & 0 & X_1 \\ 0 & -B_1 & 0 & X_2 \\ 0 & 0 & 0 & X_3 \\ X_4 & X_5 & X_6 & X_7 \end{pmatrix}$$

for some matrices X_i. Note that the diagonal blocks are all square and that the last two diagonal blocks have the same size by the above dimension calculations. This is a generalization of the kind of elementary enlargement that we considered in Proposition 3.11. Set $\mathfrak{P} = \begin{pmatrix} I_r & I_r & 0 \\ 0 & I_r & 0 \\ 0 & 0 & I_{2s} \end{pmatrix}$, where $r = |\text{cok}(i_0) \oplus \text{cok}(i_1)|$ and $s = |\frac{\text{im}(i_0)\oplus\text{im}(i_0)}{\text{im}(j)}| = |\text{im}(\partial)|$. Then

$$\mathfrak{P}'(-B_0 \boxplus B_1 \boxplus M)\mathfrak{P} = \begin{pmatrix} -B_0 & 0 & -B_0 & 0 & 0 & 0 \\ 0 & B_1 & 0 & B_1 & 0 & 0 \\ -B_0 & 0 & 0 & 0 & 0 & X_1 \\ 0 & B_1 & 0 & 0 & 0 & X_2 \\ 0 & 0 & 0 & 0 & 0 & X_3 \\ 0 & 0 & X_4 & X_5 & X_6 & X_7 \end{pmatrix}$$

contains an $r+s \times r+s$ trivial submatrix symmetric about the diagonal. Hence it is rationally null-cobordant and M is rationally cobordant to $B_0 \boxplus -B_1$, which in turn is rationally cobordant to $A_0 \boxplus -A_1$ using Proposition 3.11. The rational cobordisms become integral cobordisms by Corollary 3.8. ∎

4. Applications

In this section, we derive some consequences from the theorems of the preceding section. The first concerns the Alexander polynomials $c_n(t)$ defined in [7] and gives a result similar to that on Alexander polynomials of cobordant sphere knots [5, 13].

Corollary 4.1: *Suppose that L_0 and L_1 are disk knots $D^{2n-1} \subset D^{2n+1}$, $n > 1$, such that $\partial L_0 = \partial L_1 = K$. Then a necessary condition for L_0 and L_1 to be cobordant rel boundary is that the product of the middle-dimensional Alexander polynomials $c_n^{L_0}(t)c_n^{L_1}(t)$ be similar in $\mathbb{Q}[t,t^{-1}]$ to a polynomial of the form $p(t)p(t^{-1})$.*

Proof. By [7, §3.6] or [6, §3.6] and the calculations in Section 2 above, $c_n^{L_i}(t)$ is in the similarity class in $\mathbb{Q}[t,t^{-1}]$ of the determinant of

$$(A_i + (-1)^n A_i')^{-1}(A_i t + (-1)^n A_i'),$$

where A_i is the Seifert matrix of L_i, $i = 0,1$. We know that if L_0 and L_1 are cobordant rel boundary, then $B = A_0 \boxplus -A_1$ is rationally nullcobordant. It follows then as in [13, §15] that the determinant of $Bt + (-1)^n B'$ is similar to $\bar{p}(t)\bar{p}(t^{-1})$ for some polynomial $\bar{p}$. But clearly the determinant of $Bt+(-1)^n B'$ is equal to $\pm$ the product of the determinants of $(A_i t + (-1)^n A_i')$, $i = 0,1$. So $c_n^{L_0}(t)c_n^{L_1}(t) \sim \frac{\bar{p}(t)\bar{p}(t^{-1})}{\det(A_0+(-1)^n A_0')\det(A_1+(-1)^n A_1')}$. The claim now follows since $\frac{1}{\det(A_0+(-1)^n A_0')\det(A_1+(-1)^n A_1')}$ is a unit in $\mathbb{Q}[t,t^{-1}]$ by [7]. ∎

Since $L_0 \cup_K -L_1$ is a sphere knot $S^{2n-1} \hookrightarrow S^{2n+1}$, there is a basis for which its integral Seifert matrix A is a matrix of integers such that $A + (-1)^n A'$ is integrally unimodular. Thus each possible obstruction matrix $A_0 \boxplus -A_1$ must be cobordant to such a matrix. We can also state the following converse:

Theorem 4.2: *Let A be a matrix of integers such that $A+(-1)^n A'$ is integrally unimodular, and let L_0 be a disk knot $D^{2n-1} \subset D^{2n+1}$ with Seifert matrix A_0. Then there is a disk knot L_1 with the same boundary knot as L_0 and such* *that the obstruction Seifert matrix $A_0 \boxplus -A_1$ to L_0 and L_1 being cobordant rel boundary is cobordant to A.*

Proof. By [13], there is a sphere knot $\mathcal{K} : S^{2n-1} \hookrightarrow S^{2n+1}$ with Seifert matrix $-A$. Let L_1 be the knot $L_0 \# \mathcal{K}$, the knot sum taken away from the boundary. Then L_1 has Seifert matrix $A_0 \boxplus -A$, and $L_0 \cup_K -L_1$ has Seifert matrix cobordant to $A_0 \boxplus -A_0 \boxplus A$, which is cobordant to A. ∎

Similarly, we can show the following:

THEOREM 4.3: *Let L_0 and L_1 be two disk knots $D^{2n-1} \hookrightarrow D^{2n+1}$, $n > 1$, with common boundary K. Then there exists a sphere knot $\mathcal{K} : S^{2n-1} \hookrightarrow S^{2n+1}$ such that L_0 is cobordant to the knot sum (away from the boundary) $L_1 \# \mathcal{K}$.*

Proof. Let A_0 and A_1 be the Seifert matrices for L_0 and L_1. Then, as above, $A_0 \boxplus -A_1$ is cobordant to an integer matrix B such that $B+(-1)^n B'$ is integrally unimodular. Let $\mathcal{K}$ be a sphere knot with Seifert matrix B, which exists by [13]. Then $L_1 \# \mathcal{K}$ has Seifert matrix $A_1 \boxplus B$, and $L_0 \cup_K -(L_1 \# \mathcal{K})$ has Seifert matrix $A_0 \boxplus -A_1 \boxplus -B$, which is null-cobordant. So A_0 is cobordant to $A_1 \boxplus B$, and the theorem now follows from Theorem 3.6. ∎

Lastly, we obtain some results concerning simple disk knots.

THEOREM 4.4: *Let $K : S^{2n-2} \hookrightarrow S^{2n}$, $n > 1$ be a sphere knot. Then there is a disk knot $L : D^{2n-1} \hookrightarrow D^{2n+1}$ such that $\partial L = K$ and $\pi_i(D^{2n+1} - D^{2n-1}) \cong \pi_i(S^1)$ for $i < n$.*

Proof. By Kervaire [9, Theorem III.6], there exists some disk knot whose boundary is K (all even dimensional knots are null-cobordant). We show that in fact Kervaire's construction gives us a knot of the desired type. The argument in Kervaire's theorem proceeds as follows (modifying the notation slightly to coincide with our own): Let F be a Seifert surface for K. Then it is possible to construct a manifold V^{2n} and to embed it into D^{2n+1} such that $V \cap S^{2n-1} = F$ and $\partial V = F \cup D^{2n+1}$. This manifold V will be a Seifert surface for L, and it is obtained from F by adding handles of core dimension $\leq n$ to $F \times I$, in order of increasing dimension, to successively kill the homotopy groups of F by surgery. Then, in particular, after the addition of the 2-handles to $F \times I$, we obtain a simple connected manifold as the trace of the surgery, and ultimately $H_{2n-i}(V, F) = 0$ for $i < n$ because there are no handles of core dimension $> n$

added. Then $H_{2n-i}(V,F) \cong H^i(V)$ for $i \geq 1$, so $H^i(V) = 0$ for $1 \leq i < n$, which implies that $H_i(V) = 0$ for $1 \leq i < n$.

It now follows that $D^{2n+1} - V$ is simply-connected by the van Kampen theorem: by pushing along the bicollar of V, we can thicken V to a homotopy equivalent $2n+1$ manifold whose common boundary with the closure of its complement in D^{2n+1} is the union of two copies of V glued along L (see [11]). It then follows from the van Kampen theorem that $D^{2n+1} - V$ must be simply-connected, and from Alexander duality for a ball that $H_i(D^{2n+1} - V) = 0$ for $0 < i < n$ (see [7, Proposition 3.3] and note that these arguments extend to integer coefficients).

Now, using the usual cut-and-past construction of the infinite cyclic cover of $D^{2n+1} - L$ (see [12]), another inductive application of the van Kampen theorem shows now that the infinite cyclic cover of $D^{2n+1} - L$ is simply connected, and the Mayer–Vietoris theorem shows that its homology is trivial in dimensions $< n$. So this cover is $n-1$-connected, and it follows that the homotopy groups $\pi_i(D^{2n+1} - L)$ vanish for $1 < i < n$ and that $\pi_1(D^{2n+1} - L)) \cong \mathbb{Z}$. ∎

THEOREM 4.5: *Given any disk knot $L : D^{2n-1} \hookrightarrow D^{2n+1}$, $n > 1$, L is cobordant rel boundary to a disk knot L_1 such that $\pi_i(D^{2n+1} - L_1) \cong \pi_i(S^1)$ for $i < n$.*

Proof. First assume $n > 2$. By the preceding theorem, there exists a disk knot L_0 whose boundary agrees with that of L and which satisfies the required homotopy conditions. Let A and A_0 be the respective Seifert matrices of L and L_0. Then we know that the matrix $A \boxplus -A_0$ is cobordant to an integral matrix B such that the determinant of $B + (-1)^n B'$ is integrally unimodular since this is true for the integral Seifert matrix of the sphere knot $L \cup -L_0$. By Levine [13], there exists a sphere knot $\mathcal{K} : S^{2n-1} \subset S^{2n+1}$ whose Seifert matrix is B and such that $\pi_i(S^{2n+1} - \mathcal{K}) \cong \pi_i(S^1)$ for $i < n$. Let L_1 be the knot sum $L_0 \# \mathcal{K}$ along the interior. Then L_1 satisfies the desired homotopy properties and has Seifert matrix $A_0 \boxplus B$, which we know is cobordant to A since $A \boxplus -A_0 \boxplus -B$ is cobordant to $B \boxplus -B$, which is null-cobordant. By Theorem 3.6, L and L_1 are cobordant rel boundary.

If $n = 2$, then [13] provides a $\mathcal{K}$ only if $B + B'$ has signature a multiple of 16. But since $L \cup -L_0$ is a knot $S^2 \hookrightarrow S^4$, its Seifert matrices will all satisfy this property (again see [13]), hence so will $A \boxplus -A_0$ since signature is a matrix cobordism invariant. Thus the argument of the preceding paragraph applies again. ∎

5. Realization of cobordism classes

Up to this point we have shown that two odd-dimensional disk knots are cobordant rel boundary if and only if their Seifert matrices are cobordant. This leads to the natural question: which cobordism classes of matrices can be realized as the Seifert matrices of disk knots? In this section, we answer this question nearly completely for disk knots of sufficiently high dimension. The one obstruction to a complete classification will occur in the case of disk knots whose boundary knots have 2-torsion in their middle-dimensional Alexander modules.

First we demonstrate that we are truly dealing with a wider variety of objects than just Seifert matrices of sphere knots.

Proposition 5.1: *There exist Seifert matrices for disk knots that are not cobordant to Seifert matrices of sphere knots. In particular this implies that there are Seifert matrices for disk knots that are not cobordant to any integer matrix A such that $A+(-1)^nA'$ is integrally unimodular.*

Proof. Suppose, to the contrary, that every disk knot Seifert matrix is cobordant to some sphere knot Seifert matrix. Let us then fix a disk knot

$$L : D^{2n-1} \hookrightarrow D^{2n+1}, \quad n \text{ even}, n > 2,$$

with some Seifert surface and with Seifert matrix B. By assumption, B is cobordant to a Seifert matrix C of some sphere knot; this implies that B must have an even number of rows and columns, since this must be true of C (see, e.g., [19, p. 178]). By [13], there exists a sphere knot K with a Seifert surface that realizes the Seifert matrix $-C$. Therefore, the knot sum $K\#L$ with Seifert surface given as the boundary connected sum of the Seifert surfaces of K and L will yield the null-cobordant Seifert matrix $A = B \boxplus -C$. It then follows as in [13, §15] that the determinant of $tA + A'$ is the product of $\pm$ a power of t with a Laurent polynomial of the form $p(t)p(t^{-1})$. In particular, $|-A+A'|$ is $\pm$ a square.

Now, by [7, §3] and the calculations of Section 7, below, the middle dimensional Alexander polynomial $c_n(t)$ of a disk knot, n even, is given, up to similarity, by the determinant of $(A+A')^{-1}(At+A')$, which, with our current assumptions, must thus be of the form $\frac{p(t)p(t^{-1})}{(p(1))^2}$ (up to similarity). In particular, we see that the value $c_n(-1)$ associated to $K\#L$ must be a square. But we also know that the Alexander polynomial of a direct sum is the product of the polynomials so that $c_n^{K\#L} \sim c_n^K c_n^L$, where $\sim$ denotes similarity and we have labeled the polynomials with their knots in the obvious way. But $c_n^K(-1)$ must be $\pm$ a square since K is a sphere knot [12]. So it would follow that $c_n^L(-1)$ must also always be a square. However, this contradicts the calculations in [7, §3.64] which demonstrate that any odd number can be realized as $c_n^L(-1)$ for some L of our fixed dimension.

Hence we have demonstrated, at least for n even, that there must exist disk knot Seifert matrices that are not cobordant to sphere knot Seifert matrices. ∎

However, we do have the following proposition:

Proposition 5.2: *Suppose that A is the Seifert matrix of a disk knot $L : D^{2n-1} \hookrightarrow D^{2n+1}$ with boundary knot K. Then the matrix B is in the cobordism class of a Seifert matrix of a disk knot L' with the same boundary K if and only if $A \boxplus -B$ is cobordant to the Seifert matrix of a sphere knot $\mathfrak{K} : S^{2n-1} \hookrightarrow S^{2n+1}$.*

Proof. If L and L' are disk knots with the same boundary sphere knot K and respective Seifert matrices in the cobordism classes of A and B, then we can form the knot $\mathfrak{K} = L \cup_K -L'$ by gluing L and $-L$ together, identifying the boundaries K and $-K$. By Theorem 3.12, $A \boxplus -B$ is cobordant to the Seifert matrix of the sphere knot $\mathfrak{K}$.

Conversely, suppose that $A \boxplus -B$ is cobordant to the Seifert matrix of some sphere knot $\mathfrak{K}$. Then $-A \boxplus B$ will be the Seifert matrix of $-\mathfrak{K}$. Form $L' = L\# -\mathfrak{K}$, the internal knot sum. The Seifert matrix of this L' will be the sum of A with the Seifert matrix of $-\mathfrak{K}$ and hence will be cobordant to $A \boxplus -A \boxplus B$, which is cobordant to B. ∎

This proposition tells us how to recognize cobordism classes of Seifert matrices for disk knots with a given boundary sphere knot **provided that we already have a cobordism class of Seifert matrices with which to compare**. This is a nice start, but we would like to find a way to determine which cobordism classes are realizable starting only with information about the boundary knot. It will turn out that the crucial datum is supplied by the isometry class of the Farber–Levine torsion pairing on $T_{n-1}(\tilde{X})$, the $\mathbb{Z}$-torsion subgroup of $H_{n-1}(\tilde{X})$, so long as this group has no 2-torsion. To obtain these results, we will first establish in Theorem 5.7, below, necessary and sufficient conditions for a cobordism class of a matrix to be the cobordism class of the matrix of a disk knot. This leads to Theorem 5.10, which classifies the matrix cobordism classes of Seifert matrices for disk knots with a given boundary knot, assuming the given torsion conditions.

Let us first examine some necessary conditions for a matrix θ to be a Seifert matrix for a disk knot. We know from Section 2 that the matrix $-\theta' + (-1)^{n+1}\theta$ is the transpose of the nondegenerate intersection pairing T on $\bar{E} = \operatorname{cok}(H_n(F) \to H_n(V))/\text{torsion}$. In keeping with [7], we have also called the pairing matrix R, and to emphasize the interdependence, we will sometimes write $R = R_\theta$. We also described in Section 2 a matrix τ such that $\theta = (\tau R)'$. Since R is invertible, τ is also determined by θ as $\tau_\theta = \theta' R_\theta^{-1} = \theta'(-\theta' + (-1)^{n+1}\theta)^{-1}$. Unfortunately, we cannot simplify this further since in general θ won't be invertible.

A necessary condition then on θ is that $\tau_\theta = \theta' R_\theta^{-1} = \theta'(-\theta' + (-1)^{n+1}\theta)^{-1}$ must be integral since τ' is an integer matrix. Also, we must have $(R_\theta^{-1})'\tau_\theta R_\theta = (-\theta + (-1)^{n+1}\theta')^{-1}\theta'$ integral, since this is, up to sign, the matrix μ_θ, where μ is also as described in Section 2.

We note one implication of these requirements.

Proposition 5.3: *Let θ be the Seifert matrix of a disk knot $D^{2n-1} \hookrightarrow D^{2n+1}$. If n is odd, or if n is even and $\det(R_\theta) \neq 0$ mod 2, then θ must be even dimensional (have an even number of rows and columns).*

Proof. If n is odd, then $R_\theta = -\theta' + \theta$ is skew-symmetric. But R_θ is nondegenerate, so it must have even dimension. If n is even and $\det(R_\theta) = \det(\theta' + \theta) \neq 0$ mod 2, then also $\det(\theta' - \theta) \neq 0$ mod 2, so again θ must be even dimensional since $\theta' - \theta$ is skew symmetric and nondegenerate. ∎

We next examine the relationship between θ and the Blanchfield pairing on the cokernel of $H_n(\tilde{X}) \to H_n(\tilde{C})$ mod $\mathbb{Z}$-torsion. Once we have seen that disk knots with isometric Blanchfield pairings have cobordant Seifert matrices, we will be able to invoke some results on the realizability of Blanchfield pairings from [7] in our study of the realizability of cobordism classes of Seifert matrices. A review of the basic definitions and properties of the relevant Blanchfield pairings is presented in Section 6, below, in which we study to what extent the Blanchfield pairing of a disk knot determines its boundary knot. In this section, we take these results on Blanchfield pairings as given and complete our study of Seifert matrices.

Let us denote $\text{cok}(H_n(\tilde{X}) \to H_n(\tilde{C}))$ mod $\mathbb{Z}$-torsion by $\bar{H}$ and recall some facts from [7, §3.6] (N.B. we have altered the notation from [7] in the hopes of introducing simpler and more consistent notation). It is shown there that for a disk knot $L : D^{2n-1} \hookrightarrow D^{2n+1}$,

$$\bar{H} \otimes \mathbb{Q} \cong H_n(\tilde{C}; \mathbb{Q}) / \ker(p : H_n(\tilde{C}; \mathbb{Q}) \to H_n(\tilde{C}, \tilde{X}; \mathbb{Q}))$$

is presented as a $\Gamma = \mathbb{Q}[\mathbb{Z}]$-module by the matrix $(-1)^{n+1}(R^{-1})'\tau Rt - \tau'$. Recall that these matrices are given in terms of bases of the **integer** homology groups of the Seifert surface. Also with respect to these integral bases (which induce a basis for $\bar{H} \otimes \mathbb{Q}$), the matrix of the self-Blanchfield pairing on $\bar{H}$ is given by $\frac{t-1}{(R^{-1})'\tau - (-1)^{n+1} t\tau' R^{-1}}$. Using the integrality of the appropriate bases, it is not hard to see from [7, §3.6] that the same matrix $M = (-1)^{n+1}(R^{-1})'\tau Rt - \tau'$ in fact presents $\bar{H}$ as a Λ-module, and the matrix $\frac{t-1}{(R^{-1})'\tau - (-1)^{n+1} t\tau' R^{-1}}$ also represents the **integral** Blanchfield pairing $\bar{H} \otimes \bar{H} \to Q(\Lambda)/\Lambda$.

We will prove that Seifert matrices of disk knots with isometric Blanchfield pairings are cobordant.

Theorem 5.4: *Let θ_1 and θ_2 be Seifert matrices for disk knots*

$$L_1, L_2 : D^{2n-1} \hookrightarrow D^{2n+1}.$$

Suppose that L_1 and L_2 have isometric Blanchfield self-pairings on $\bar{H}_1 \cong \bar{H}_2$. Then θ_1 and θ_2 are cobordant, in fact rationally S-equivalent.

Proof. The proof will require a few lemmas.

By using some slightly modified machinery of Trotter [19], we will actually work mostly with the rational Blanchfield pairing, which will simplify things considerably. For one thing, we can note that $M = (-1)^{n+1}(R^{-1})'\tau Rt - \tau' = (R^{-1})'((-1)^{n+1}\theta' t - \theta)$ and $\frac{t-1}{(R^{-1})'\tau - (-1)^{n+1} t\tau' R^{-1}} = R\frac{1-t}{\theta' - (-1)^{n+1} t\theta} R'$. So, we see that up to a change of basis over $\mathbb{Q}$, the presentation matrix and Blanchfield pairing matrix of the Γ-module $\bar{H} \otimes \mathbb{Q}$ are simply $((-1)^{n+1}\theta' t - \theta)$ and $\frac{1-t}{\theta' - (-1)^{n+1} t\theta}$.

We will also need the notion of rational S-equivalence. For two square rational matrices A and B, we say that A is a rational row enlargement of B and B is a rational row reduction of A if

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & x & u \\ 0 & v & B \end{pmatrix},$$

where x and 1 are rational numbers, v is a column vector, and everything else is made to make the matrix square. Rational column enlargements and reductions are defined similarly with the transposed form. Rational S-equivalence is then the equivalence relation generated by rational row and column enlargements and reductions and by rational congruence. Two rationally S-equivalent matrices are rationally cobordant by the same arguments as in Proposition 3.11.

We will need the following basic lemma.

LEMMA 5.5: *For any disk knot Seifert matrix θ, either θ is rationally S-equivalent to a rationally nonsingular matrix or $(-1)^{n+1}\theta' t - \theta$ presents the Γ-module* 0.

Proof. The proof can be obtained from minor modifications to the proof of [19, Lemma 1.4], using also minor modifications of the work on pages 484–485 of [18]. ∎

We will see in the next lemma that two rationally S-equivalent matrices present the same Γ-module.

LEMMA 5.6: *If θ_1 and θ_2 are rationally S-equivalent, then they determine isometric Γ-modules with self-Blanchfield pairings.*

Proof. The proofs of Lemmas 1.4 and 1.2 of [19] apply rationally. It should be noted that our presentation matrix and pairing matrix defer slightly from those in [19]. One reason is that we employ a different convention for turning a matrix into a pairing matrix (we use $a_1 \times a_2 \to a_1' M \bar{a}_2$, while Trotter uses $a_1 \times a_2 \to \bar{a}_2' M a_1$). The other difference is the appearance of $\frac{1}{t-1}$ in Trotter's presentation matrices, but, as noted in [19, p. 179], these make no difference as multiplication by $t-1$ is an automorphism of knot modules. So the translation to Trotter's algebraic language from the topological language can be made via some isomorphisms and convention switches, and so his results apply to our case. (One should also note carefully that what he calls Λ is our $\mathbb{Z}[t, t^{-1}, (1-t)^{-1}]$, while our Λ is denoted there by Λ_0.) ∎

We now need to consider Trotter's **trace** function [19]: Since the rational functions, i.e. elements of $Q(\Lambda)$, can be written in terms of partial fractions, $Q(\Lambda)$ splits over $\mathbb{Q}$ into the direct sum of $\Gamma[(1-t)^{-1}]$ and the subspace P consisting of 0 and proper fractions with denominators prime to t and $1-t$. The trace χ is then defined as the $\mathbb{Q}$-linear map to $\mathbb{Q}$ determined by $\chi(f) = f'(1)$ if $f \in P$ and 0 if $f \in \Gamma[(1-t)^{-1}]$. The $'$ here denotes derivative with respect to t. This then induces a map $Q(\Gamma)/\Gamma \cong Q(\Lambda)/\Gamma \to \mathbb{Q}$. In particular, by composing χ with the Blanchfield pairing, one obtains a rational scalar form $(\bar{H} \otimes \mathbb{Q}) \times (\bar{H} \otimes \mathbb{Q}) \to \mathbb{Q}$.

It is clear that two Seifert matrices that induce isometric Blanchfield forms induce isometric rational scalar forms.

Now by [19, Lemma 2.7b], for $f \in P$, $\chi((t-1)f) = f(1)$. And also, as in [19, Lemma 2.10], Δ has degree equal to the dimension of θ and nonzero constant term, plus we know it is prime to $(t-1)$, so by Cramer's rule, each term in $(\theta' - (-1)^{n+1}t\theta)^{-1}$ lies in P. Thus χ applied to $\frac{t-1}{\theta'-(-1)^{n+1}t\theta}$ is give by evaluation of $\frac{1}{\theta'-(-1)^{n+1}t\theta}$ at 1, so we just get $\frac{1}{\theta'-(-1)^{n+1}\theta}$ as the matrix of the rational scalar pairing.

It follows as in the proof of [19, 2.11], using [19, 2.5 and 2.10], which also hold rationally, that a choice of $\mathbb{Q}$-vector space basis in an isometry class of a finitely generated $\Lambda[(t-1)^{-1}]$-module H_0 with a rational scalar form **determines** a "Seifert matrix" θ_0 and that our given $\bar{H} \otimes \mathbb{Q}$ with rational scalar form is isometric to H_0 if and only if there is a basis for $\bar{H} \otimes \mathbb{Q}$ with respect to which its Seifert matrix θ is equal to θ_0: The existence of an isometry implies that there are bases with respect to which both scalar forms have the same matrix S of [19], and, with respect to these bases, $(1-t)^{-1}$ acts by the same matrix γ, but then the equations in [19] determine both θ_0 and θ by γS^{-1}. Finally, by [19, Proposition 2.12], this implies that two rationally nonsingular Seifert matrices determine isometric rational scalar forms if and only if they are rationally congruent.

We can now complete the proof of Theorem 5.4. By hypothesis θ_1 and θ_2 determine isometric Blanchfield forms, hence they induce isometric scalar forms. Furthermore, by Lemma 5.5, θ_1 and θ_2 are rationally S-equivalent to Seifert forms, say $\hat{\theta}_1$ and $\hat{\theta}_2$, respectively, that are rationally nonsingular and which, by Lemma 5.6, still determine isometric scalar forms. By the immediately preceding discussion, $\hat{\theta}_1$ and $\hat{\theta}_2$ are rationally congruent. It follows that θ_1 and

θ_2 are rationally S-equivalent and hence, in particular, cobordant as seen in the proof of Proposition 3.11.

This completes the proof of Theorem 5.4. ∎

The relationships we have just established between Seifert matrices and Blanchfield pairings turn out to be just what we need to realize rational cobordism classes of Seifert matrices.

THEOREM 5.7: *Let θ be any square matrix satisfying the necessary conditions to be the integral Seifert matrix of a disk knot $D^{2n-1} \hookrightarrow D^{2n+1}$, i.e., such that*

1. *$R_\theta = -\theta' + (-1)^{n+1}\theta$ is nondegenerate, and*
2. *$\tau_\theta = \theta'(-\theta' + (-1)^{n+1}\theta)^{-1}$ and $\mu_\theta = (-\theta + (-1)^{n+1}\theta')^{-1}\theta'$ are integral matrices.*

Then for any $n > 2$, there is a disk knot $D^{2n-1} \hookrightarrow D^{2n+1}$ whose Seifert matrix is cobordant to θ.

Proof. Given such a θ, it determines a Λ-module $\bar{H}$ with a $(-1)^{n+1}$-Hermitian pairing to $Q(\Lambda)/\Lambda$ by the matrices

$$(-1)^{n+1}(R^{-1})'\tau Rt - \tau' \quad \text{and} \quad \frac{t-1}{(R^{-1})'\tau - (-1)^{n+1}t\tau' R^{-1}}$$

as in the discussion earlier in this section (see also [7, §3.6.3]). Note that $\bar{H}$ is $\mathbb{Z}$-torsion free by the same arguments as in [19, Lemma 2.1]. By [7, Proposition 3.21], there exists a simple disk knot L realizing this module and pairing with $\bar{H} = H_n(\tilde{C})$ and also with simple boundary knot such that $H_{n-1}(\tilde{X})$ is $\mathbb{Z}$-torsion. By Theorem 5.4, any Seifert matrix for L is cobordant to our given θ; in fact it is rationally S-equivalent to it. ∎

So, at this point we have demonstrated that, for $n > 2$, every allowable cobordism class can be realized by 1) showing that a potential Seifert matrix determines a Blanchfield pairing, 2) constructing every possible Blanchfield pairing, and 3) showing that a Blanchfield pairing determines its Seifert matrices up to rational S-equivalence. So by constructing every possible pairing, we construct every possible cobordism class. However, we have not said anything yet about what boundary knots we get. The constructions of Theorem 5.7 give only simple disk knots whose boundaries are simple sphere knots and such that $H_{n-1}(\tilde{X})$ is $\mathbb{Z}$-torsion (this follows from the construction in [7, Proposition 3.21] and the construction in [15, §12] that it is modeled after). Such sphere knots are called

finite simple. In this special case, we can say a lot immediately. We will show in Section 6 below that in this situation the Blanchfield pairing on $H_n(\tilde{C})$ completely determines the Farber–Levine torsion pairing on $H_{n-1}(\tilde{X})$. In fact, we will prove the following

THEOREM 5.8 (Corollary 6.3): *For a simple disk knot $L : D^{2n-1} \hookrightarrow D^{2n+1}$, the Λ-module $T_{n-1}(\tilde{X})$ and its Farber–Levine $\mathbb{Z}$-torsion pairing are determined up to isometry by any Seifert matrix for L.*

In this situation, we will say that **the Seifert matrix induces the Farber–Levine pairing**.

We can now apply the following theorem of Kojima [10] (which we have translated into our notation):

THEOREM 5.9 (Kojima): *Suppose that K_0 and K_1 are two finite simple sphere knots $S^{2n-2} \to S^{2n}$, $n \geq 5$, $H_{n-1}(\tilde{X}_0) \cong H_{n-1}(\tilde{X}_1)$ contains no 2-torsion, and the Farber-Levine pairings on $H_{n-1}(\tilde{X}_0)$ and $H_{n-1}(\tilde{X}_1)$ are isometric, then K_0 and K_1 are isotopic knots.*

Putting this theorem together with the results of Section 6, quoted above, we see that, for $n \geq 5$, the following statement holds: if a Blanchfield pairing on $H_n(\tilde{C})$ induces a $T_{n-1}(\tilde{X})$ with no 2-torsion, then this Blanchfield pairing determines a unique finite simple sphere knot $S^{2n-2} \hookrightarrow S^{2n}$ which must be the boundary knot of any simple disk knot possessing this Blanchfield pairing and having a finite simple boundary knot. In particular since Seifert matrices determine Blanchfield pairings, the Seifert matrix of a simple disk knot with finite simple boundary knot determines the boundary knot uniquely, so long as $H_{n-1}(\tilde{X}) = T_{n-1}(\tilde{X})$ has no 2-torsion.

We can now immediately generalize this to prove the following theorem about realizability of cobordism classes of Seifert matrices for more arbitrary boundary knots:

THEOREM 5.10: *Let $K : S^{2n-2} \hookrightarrow S^{2n}$, $n \geq 5$, be a sphere knot with complement X such that $T_{n-1}(\tilde{X})$ contains no 2-torsion. Then there exists a disk knot $L : D^{2n-1} \hookrightarrow D^{2n+1}$ with boundary knot K and with Seifert matrix in a given cobordism class $[\theta]$ if and only if there is an integral matrix θ in the class such that*

1. *$R_\theta = -\theta' + (-1)^{n+1}\theta$ is nondegenerate,*

2. *$\tau_\theta = \theta'(-\theta' + (-1)^{n+1}\theta)^{-1}$ and $\mu_\theta = (-\theta + (-1)^{n+1}\theta')^{-1}\theta'$ are integral matrices, and*
3. *the Farber–Levine pairing induced by θ is isometric to the Farber–Levine pairing on $T_{n-1}(\tilde{X})$.*

Proof. Suppose we have such a knot L and its cobordism class of Seifert matrices $[\theta]$. We show that there is a Seifert matrix in the cobordism class satisfying the listed properties: We know that the first two requirements are always necessary for a Seifert matrix. For the third, recall that by Theorem 4.5, any disk knot is cobordant rel boundary to a simple disk knot, and by Theorem 3.6, any two such disk knots have cobordant Seifert matrices. By Theorem 5.8, any Seifert matrix of a simple disk knot determines the Farber–Levine pairing on $T_{n-1}(\tilde{X})$ of the boundary knot up to isometry. So there is a Seifert matrix in the cobordism class $[\theta]$ that induces the correct Farber–Levine pairing (up to isometry).

Conversely, given a θ that meets the above requirements, Theorem 5.7 and its proof assure us that we can construct a simple disk knot L_1 with finite simple boundary whose Seifert matrices fall in the cobordism class $[\theta]$ of θ and induce the given Farber–Levine pairing on the boundary knot. Now let L_0 be any simple disk knot with our given K as boundary. Such a knot always exists since K is null-cobordant by its dimensions and [9], and there is a cobordism rel boundary of any disk knot to a simple disk knot by Theorem 4.5. Let θ_0 be any Seifert matrix of L_0, and note that θ_0 determines the Farber–Levine pairing on $T_{n-1}(\tilde{X})$. Also, again by Theorem 5.7, there is a simple disk knot with torsion simple boundary θ_0 whose Seifert matrices fall in the cobordism class $[\theta_0]$ and induce the given Farber–Levine pairing. Since L_1 and L_0 are both simple disk knots with torsion simple boundaries K_1 and K_0 and since the boundary modules $H_{n-1}(\tilde{X}_0)$ and $H_{n-1}(\tilde{X}_1)$ are Farber–Levine isometric by construction and contain no 2-torsion by assumption, Kojima's Theorem [10] implies that K_0 and K_1 are isometric. So now let us form the sphere knot $\mathcal{K} = L_1 \cup_{K_0} -L_0$. By Theorem 3.12, the Seifert matrix of $\mathcal{K}$ is cobordant to $\theta \boxplus -\theta_0$. Finally, we form the connected sum away from the boundary $L = L_0 \# \mathcal{K}$. Then L has Seifert matrix cobordant to $\theta_0 \boxplus (\theta \boxplus -\theta_0) = \theta$, and it is our desired knot. ∎

We note that the statement of the theorem only guarantees that some element in the cobordism class determines the proper Farber–Levine pairing, not all elements. This is really the best that can be hoped for since given an arbitrary disk knot, it is possible that $T_{n-1}(\tilde{X})$ may not be in the image of ∂_*

or there may be elements in $T_{n-1}(\tilde{X})$ that are in the image of $T_n(\tilde{C}, \tilde{X})$. The Farber–Levine pairing on such elements clearly won't be determined by the Seifert matrix. However, as noted in the proof, there is always a cobordism rel boundary to a simple disk knot for which the entirety of the Farber–Levine pairing **is** determined by the Seifert matrix, and we know that such a cobordism keeps the Seifert matrix in its cobordism class. While this argument shows that a cobordism class does not determine a Farber–Levine pairing, we make the following conjecture:

Conjecture: The cobordism class of any integer matrix satisfying

1. $R_\theta = -\theta' + (-1)^{n+1}\theta$ is nondegenerate,
2. $\tau_\theta = \theta'(-\theta' + (-1)^{n+1}\theta)^{-1}$ and $\mu_\theta = (-\theta + (-1)^{n+1}\theta')^{-1}\theta'$ are integral matrices

determines a unique element in the Witt group of $\mathbb{Z}$-linear conjugate self-adjoint $(-1)^{n+1}$-symmetric nonsingular pairings to $\mathbb{Q}/\mathbb{Z}$ on finite $\Lambda[(t-1)^{-1}]$-modules.

Our realization theorem makes no conclusions about knots for which $T_{n-1}(\tilde{X})$ possesses 2-torsion. This is because finite simple even-dimensional sphere knots are not determined entirely by their Farber–Levine pairings, and so the previous proof breaks down; we cannot apply the theorem of Kojima. It was shown by Farber in a series of papers culminating in [3, 4] (see also [2]) that in this case there is also an even-torsion pairing on the stable homotopy groups $\sigma_{n+1}(\tilde{X})$ that plays a role in the classification. In fact, Farber shows that such knots are classified completely by the algebraic invariants in their **Λ-quintets**. It remains unclear whether the Seifert matrices and/or Blanchfield pairings of a simple disk knot are sufficient to determine the Λ-quintets of their boundary knots, so we cannot yet broaden Theorem 5.10 to include realizability for all knots. An alternative procedure would be to show that all knots constructed in Theorem 5.7 that give the same Farber–Levine pairing on the boundary just happen to have the same actual boundary knot. If so, the proof of Theorem 5.10 would apply without the need to invoke a broader classification theorem. However, we have not yet been able to establish this either.

6. Blanchfield pairings determine Farber–Levine pairings

In this section, we will establish that for a simple disk knot of odd dimension $D^{2n-1} \hookrightarrow D^{2n+1}$, the Farber–Levine $\mathbb{Z}$-torsion self-pairing of the boundary knot, $T_{n-1}(\tilde{X}) \otimes T_{n-1}(\tilde{X}) \to \mathbb{Q}/\mathbb{Z}$, is determined completely by the module $\bar{H} = \text{cok}(H_n(\tilde{X}) \to H_n(\tilde{C}))/(\mathbb{Z}\text{-torsion})$ and its self-Blanchfield pairing. This result was used in the previous section in conjunction with the main theorem of [10] to recognize the boundary knots of disk knots we have constructed.

We will begin by demonstrating that the module $H_{n-1}(\tilde{X})$ and the Farber–Levine pairing on its Z-torsion submodule $T_{n-1}(\tilde{X})$ are determined by the self-Blanchfield pairing on $H_n(\tilde{C})$. These pairings will be defined in detail below.

Theorem 6.1: *Given a simple disk knot $D^{2n-1} \hookrightarrow D^{2n+1}$, the module $H_{n-1}(\tilde{X})$ and the Farber–Levine $\mathbb{Z}$-torsion pairing on $T_{n-1}(\tilde{X})$ are determined up to isometry by the isometry class of the Blanchfield self-pairing on $H_n(\tilde{C})$.*

In the proof we develope a formula relating the two pairings based upon the geometry of chains. This will allow us to prove that the isometry class of the latter completely determines the isometry class of the former. We then show that, in fact, $\bar{H}$, which algebraically corresponds to the quotient of $H_n(\tilde{C})$ by its annihilating submodule, is sufficient to determine $T_{n-1}(\tilde{X})$ and its Farber–Levine pairing.

We first undertake some preliminary work.

To simplify things marginally, observe that $\partial\tilde{C} = \tilde{X} \cup_{S^{2n-2}\times\mathbb{R}} D^{2n-1} \times \mathbb{R}$ so that, for $n \geq 2$, the map induced by inclusions $H_{n-1}(\tilde{X}) \to H_{n-1}(\partial\tilde{C})$ is an isomorphism and $H_n(\tilde{X}) \to H_n(\tilde{C})$ is an epimorphism. It therefore follows from the five lemma applied to the exact sequences of the pairs that $H_n(\tilde{C}, \tilde{X}) \to H_n(\tilde{C}, \partial\tilde{C})$ is an isomorphism. For $n = 1$, $X \sim_{h.e.} S^1$, so $\tilde{X} \sim_{h.e.} *$. In this case there is no Farber–Levine pairing of interest, so we will shall always assume $n \geq 2$. We will work with $\partial\tilde{C}$ or $\tilde{X}$ as convenient, but using these isomorphisms, we can assume that all relevant chains are actually contained in $\tilde{X}$.

For a simple disk knot $D^{2n-1} \hookrightarrow D^{2n+1}$, $H_i(\tilde{C}) = 0$ for $0 < i < n$ due to the connectivity assumptions. Now, as observed in [15] (and holding for any regular covering of a compact piecewise-linear n-manifold with boundary), $H_*(\tilde{C}) \cong \overline{H_e^{2n+1-*}(\tilde{C}, \partial\tilde{C})}$, the conjugate of the cohomology of the cochain complex $\text{Hom}_\Lambda(C_*(\tilde{C}, \partial\tilde{C}), \Lambda)$. Similarly, $H_*(\tilde{C}, \partial\tilde{C}) \cong \overline{H_e^{2n+1-*}(\tilde{C})}$, the conjugate of the cohomology of the cochain complex $\text{Hom}_\Lambda(C_*(\tilde{C}), \Lambda)$. It now follows from Proposition 2.4 of [15] and this generalization of Poincaré duality that there exist short exact sequences

$$0 \longrightarrow \operatorname{Ext}^2_\Lambda(H_{n-2}(\tilde{C}),\Lambda) \longrightarrow \overline{H_{n+1}(\tilde{C},\partial\tilde{C})} \longrightarrow \operatorname{Ext}^1_\Lambda(H_{n-1}(\tilde{C}),\Lambda) \longrightarrow 0$$

$$0 \longrightarrow \operatorname{Ext}^2_\Lambda(H_{n-1}(\tilde{C}),\Lambda) \longrightarrow \overline{H_{n}(\tilde{C},\partial\tilde{C})} \longrightarrow \operatorname{Ext}^1_\Lambda(H_{n}(\tilde{C}),\Lambda) \longrightarrow 0.$$

By the connectivity assumptions on $\tilde{C}$, these imply that $H_{n+1}(\tilde{C},\partial\tilde{C}) = H_{n+1}(\tilde{C},\tilde{X}) = 0$, and $\overline{H_n(\tilde{C},\partial\tilde{C})} \cong \operatorname{Ext}^1_\Lambda(H_n(\tilde{C}),\Lambda)$. Since $H_n(\tilde{C})$ is of type K (it is finitely generated and $t-1$ acts as an automorphism), $\operatorname{Ext}^1_\Lambda(H_n(\tilde{C}),\Lambda)$ is $\mathbb{Z}$-torsion free by [15, Proposition 3.2], hence so is $H_n(\tilde{C},\partial\tilde{C}) \cong H_n(\tilde{C},\tilde{X})$.

So we have shown that there is an exact sequence of Alexander modules

$$0 \longrightarrow H_n(\tilde{X}) \longrightarrow H_n(\tilde{C}) \xrightarrow{p_*} H_n(\tilde{C},\tilde{X}) \xrightarrow{\partial_*} H_{n-1}(\tilde{X}) \longrightarrow 0$$

and that $H_n(\tilde{C},\tilde{X})$ has no $\mathbb{Z}$-torsion. We seek first to determine how the self-Blanchfield pairing on $H_n(\tilde{C})$ determines the Farber–Levine $\mathbb{Z}$-torsion pairing $[\,,\,] : T_{n-1}(\tilde{X}) \otimes T_{n-1}(\tilde{X}) \to \mathbb{Q}/\mathbb{Z}$, where $T_{n-1}(\tilde{X})$ is the $\mathbb{Z}$-torsion subgroup of $H_{n-1}(\tilde{X})$.

Now we recall the definitions of the various pairings involved. The following discussion integrates the relevant work from papers of Blanchfield [1] and Levine [15] and adapts it, where necessary, to the case of disk knots.

We can assume that $\tilde{C}$, the infinite cyclic cover of the exterior of the disk knot $L : D^{N-2} \hookrightarrow D^N$, is triangulated equivariantly so that $C_*(\tilde{C},\partial\tilde{C})$ is a free left Λ-module with basis given by the cells of C not in ∂C. Then $C_*(\tilde{C})$ can be taken as the free left Λ-module with basis given by the dual cells to the given triangulation of C [15]. One then defines an intersection pairing of left Λ-modules to Λ at the chain level by setting $a \cdot b = \sum_i S(a, t^i b)t^i$ for $a \in C_i(\tilde{C})$, $b \in C_{N-i}(\tilde{C},\partial\tilde{C})$, where S is the ordinary intersection pairing of chains. The pairing $\cdot$ descends to a well-defined pairing of homology modules. It also follows from the properties of the ordinary intersection form on a manifold that if $x \in H_i(\tilde{C})$ and $y \in H_{N-i}(\tilde{C})$, then $x \cdot p_*(y) = (-1)^{i(N-i)}\overline{y \cdot p_*(x)}$, where $p_* : H_n(\tilde{C}) \to H_n(\tilde{C},\partial\tilde{C})$.

From here, it is possible to define a linking pairing, the **Blanchfield pairing**, $V : W_i(\tilde{C}) \otimes W_{N-1-i}(\tilde{C},\partial\tilde{C}) \to Q(\Lambda)/\Lambda$, where $W_i(X)$ is the submodule of weak boundaries of $C_i(X)$, i.e., those chains c such that λc bounds for some $\lambda \in \Lambda$, and $Q(\Lambda)$ is the field of rational functions. If $a \in W_i(\tilde{C})$, $b \in W_{N-1-i}(\tilde{C},\partial\tilde{C})$, and $A \in C_{i+1}(\tilde{C})$ with $\partial A = \alpha a$ for some $\alpha \in \Lambda$, then $V(a,b) = \frac{1}{\alpha}A \cdot b$ by definition. Note that this linking number is well-defined to $Q(\Lambda)$ at the chain level. However, in order to descend to a well-defined map on homology classes with torsion, it is necessary to consider the image of V in $Q(\Lambda)/\Lambda$. In the case of interest to us, the relevant pairing will be $V : H_n(\tilde{C}) \otimes H_n(\tilde{C}, \partial\tilde{C}) \to Q(\Lambda)/\Lambda$ when $N = 2n+1$ (recall that both modules are Λ-torsion so all cycles weakly bound). By [15, §5], since $H_n(\tilde{C}, \partial\tilde{C})$ is $\mathbb{Z}$-torsion free, the pairing is nonsingular in the sense that its adjoint provides an isomorphism $\overline{H_n(\tilde{C}, \partial\tilde{C})} \to \mathrm{Hom}_\Lambda(H_n(\tilde{C}), Q(\Lambda)/\Lambda)$ (the overline on $\bar{H}_n(\tilde{C}, \partial\tilde{C})$ indicates that we take the module with the conjugate action of Λ under the antiautomorphism $t \to t^{-1}$, reflecting the fact that V will be conjugate linear, since $\cdot$ is). This pairing determines a self-pairing $\langle\, , \rangle$ on $H_n(\tilde{C})$ by $\langle a, b\rangle = V(a, p_*(b))$. This pairing is $(-1)^{n+1}$-Hermitian, i.e., $\langle a, b\rangle = (-1)^{n+1}\overline{\langle b, a\rangle}$, and it is nondegenerate on $\mathrm{coim}(p_*)$.

It requires more work to define the Farber–Levine $\mathbb{Z}$-torsion pairing. These are pairings $[\, ,] : T_i(\tilde{X}) \otimes T_{N-i-2}(\tilde{X}) \to \mathbb{Q}/\mathbb{Z}$, where $T_j(\tilde{X})$ is the $\mathbb{Z}$-torsion submodule of $H_j(\tilde{X})$ and X has dimension N. We will specialize immediately to our case of interest where $N = 2n$, $i = n-1$, and $T_{n-1}(\tilde{X})$ is the the torsion Alexander module of the boundary knots of a simple disk knot. In [15], Levine begins with a sophisticated definition via homological algebra and then produces an equivalent geometric formulation. We will be more concerned with the geometric formulation, but there is one intermediate algebraic construction that remains necessary.

We first need to choose two integers, though the final outcome will be independent of the choice modulo the restrictions on choosing. Let m be a positive integer such that $mT_e^{n+1}(\tilde{X}) = 0$, where $T_e^{n+1}(\tilde{X})$ is the torsion subgroup of $H_e^{n+1}(\tilde{X})$. By generalized Poincaré duality, $T_e^{n+1}(\tilde{X}) \cong \overline{T_{n-1}(\tilde{X})}$, so m kills this module as well. Such an m exists since $T_{n-1}(\tilde{X})$ is finite by [15, Lemma 3.1]. Next, let $\Lambda_m = \Lambda/m\Lambda = \mathbb{Z}_m[\mathbb{Z}]$, and let $\theta = \Lambda/(t^k - 1)$, where k is a positive integer chosen large enough so that $t^k - 1$ annihilates $H_e^n(\tilde{X}; \Lambda_m)$. Such a k exists since $H_e^n(\tilde{X}; \Lambda_m) \cong \overline{H_n(\tilde{X}; \Lambda_m)}$ by generalized Poincaré duality, and this module is also finite, again by [15, Lemma 3.1] and the argument on the bottom of page 18 of [15]. Since $H_e^n(\tilde{X}; \Lambda_m)$ is finite and t acts isomorphically, $t^k = 1$ for some integer $k > 0$. Hence $t^k - 1$ annihilates the module for this choice of k. Note that $t^k - 1$ also kills $H_n(\tilde{X}; \Lambda_m)$ since $\overline{t^k - 1} = t^{-k} - 1 = -t^{-k}(t^k - 1)$ and t acts automorphically.

By the same arguments, we can find a k such that $t^k - 1$ annihilates $H_n(\tilde{C}; \Lambda_m)$. Since $H_{n+1}(\tilde{C}, \tilde{X}) = 0$, we also get $H_n(\tilde{C}, \tilde{X}; \Lambda_m) = 0$ because, as an abelian group, $H_{n+1}(\tilde{C}, \tilde{X}; \Lambda_m) = H_{n+1}(\tilde{C}, \tilde{X}) \otimes_{\mathbb{Z}} \mathbb{Z}_m$ (recall that $H_n(\tilde{C}, \tilde{X})$ is $\mathbb{Z}$-torsion free). So $H_n(\tilde{X}; \Lambda_m)$ maps monomorphically into $H_n(\tilde{C}; \Lambda_m)$ in the long exact sequence of the pair $(\tilde{C}, \tilde{X})$ with Λ_m coefficients, so this k suffices to kill $H_n(\tilde{X}; \Lambda_m)$ as well. In other words, for any k such that $t^k - 1$ kills $H_n(\tilde{C}; \Lambda_m)$, the same choice of k gives a $t^k - 1$ that also kills $H_n(\tilde{X}; \Lambda_m)$.

The geometric part of the construction now finds a pairing $\{\, , \} : T_{n-1}(\tilde{X}) \otimes T_{n-1}(\tilde{X}) \to I(\theta)/\theta$, where $I(\theta)$ is the Λ-injective envelope of θ. But for a finite Λ-module A, $\mathrm{Hom}_\Lambda(A, I(\theta)/\theta) = \mathrm{Hom}_\Lambda(A, (\mathbb{Q} \otimes_{\mathbb{Z}} \theta)/\theta) \overset{e}{\cong} \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$, so it is possible to define $[\, ,]$ as the composition of $\{\, , \}$ with these isomorphisms.

The pairing $\{\, , \}$ can be described in the following geometric manner: Suppose that z and w are cycles representing elements of $T_{n-1}(\tilde{X})$. By the choice of m, mz is null-homologous, so $mz = \partial z'$ for some $z' \in C_n(\tilde{X})$. Then $(t^k - 1)z'$ is null-homologous mod m since we know that $t^k - 1$ annihilates $H_n(\tilde{X}; \Lambda_m)$. Thus, we can write $(t^k - 1)z' = \partial z'' + mz_0$ for some $z'' \in C_{n+1}(\tilde{X})$ and $z_0 \in C_n(\tilde{X})$. Then one sets $\{z, w\}$ to be the image of $(-z'' \cdot w)/m$, which is in $\Gamma = \Lambda \otimes \mathbb{Q}$, under the composition $\Gamma \to \mathbb{Q} \otimes \theta \subset I(\theta) \to I(\theta)/\theta$. It turns out that this pairing is independent of the choices involved and descends to a well-defined map on the homology torsion subgroups. See [15] for more details.

Using this geometric definition, we will prove that the middle dimensional pairing $\{\, , \} : T_{n-1}(\tilde{X}) \otimes T_{n-1}(\tilde{X}) \to I(\theta)/\theta$ can be expressed in terms of the linking pairing $V : H_n(\tilde{C}) \otimes H_n(\tilde{C}, \partial\tilde{C})$. In the following computations, all pairings are defined at the chain level, so there is no ambiguity. Consider cycles z, w representing elements in $T_{n-1}(\tilde{X})$. Let $z', z'', z_0 \in C_*(\tilde{X})$ be as defined above. We need to reformulate $(-z'' \cdot w)/m$, where the intersection product is that in $\tilde{X}$.

Since $\partial_* : H_n(\tilde{C}, \tilde{X}) \to H_{n-1}(\tilde{X})$ is surjective, there exist chains $X, Y \in C_n(\tilde{C})$ such that $\partial X = z$ and $\partial Y = w$. Then, the intersection number $z'' \cdot w$ in $\tilde{X}$ is equal to the intersection number of z'' and Y in $\tilde{C}$. This follows just as in the more standard case of intersection numbers for manifolds with boundary.

Next, observe that $0 = \partial^2 z'' = \partial((t^k - 1)z' - mz_0) = (t^k - 1)mz - m\partial z_0$, which implies that $\partial z_0 = (t^k - 1)z$. This also implies the important fact that $t^k - 1$ annihilates $T_{n-1}(\tilde{X})$ since z is an arbitrary element of it. Let $S = (t^k - 1)X - z_0 \in C_n(\tilde{C})$. The chain S is a cycle and so represents an element of $H_n(\tilde{C})$. Since $H_n(\tilde{C})$ is a finitely generate Λ-torsion module, there exists an element $\Delta \in \Lambda$ such that $\Delta H_n(\tilde{C}) = 0$. So there exists a chain $R \in C_{n+1}(\tilde{C})$ such that $\partial R = \Delta S$. Similarly, define the n-cycle $B = mX - z'$, and choose an $n+1$ chain A in $\tilde{C}$ such that $\partial A = \Delta B$.

Now $\partial(mR - (t^k - 1)A) = m\partial R - (t^k - 1)\partial A = m\Delta S - (t^k - 1)\Delta B = \Delta(m(t^k - 1)X - mz_0 - (t^k - 1)(mX - z')) = \Delta\partial z''$. Using the properties of intersection forms we can see that $\Delta z'' \cdot Y = (mR - (t^k - 1)A) \cdot Y$. In fact, $(\Delta z'' - mR + (t^k - 1)A)$ is a cycle in $C_{n+1}(\tilde{C})$ and so represents a homology class. Thus $(\Delta z'' - mR + (t^k - 1)A) \cdot Y$ is a well-defined element of Λ under the intersection pairing $H_{n+1}(\tilde{C}) \otimes H_n(\tilde{C}, \tilde{X}) \to \Lambda$. But we know this pairing is Λ-linear in $H_{n+1}(\tilde{C})$, and $H_{n+1}(\tilde{C})$ is Λ-torsion. So this intersection must be $0 \in \Lambda$.

Thus, since the intersection $z'' \cdot w$ in $\tilde{X}$ is equal to the intersection number $z'' \cdot Y$ in $\tilde{C}$, we compute

$$\begin{aligned} \frac{z'' \cdot_{\tilde{X}} w}{m} = \frac{z'' \cdot_{\tilde{C}} Y}{m} &= \frac{(mR - (t^k - 1)A) \cdot Y}{m\Delta} \\ &= \frac{R \cdot Y}{\Delta} - \frac{t^k - 1}{m}\frac{A \cdot Y}{\Delta} \\ &= V(S, Y) - \frac{t^k - 1}{m} V(B, Y), \end{aligned}$$

and this establishes a formula for $\{z, w\} = \frac{-z'' \cdot_{\tilde{X}} w}{m}$ in terms of the linking pairing V under the projection to $I(\theta)/\theta$. Note that this formula is well-defined on passage to homology, since we know in this case that V is well-defined up to elements of Λ. So the first term of this expression is well-defined up to an element of Λ and the second term up to elements of the form $\frac{t^k-1}{m}\lambda$, $\lambda \in \Lambda$. But all such elements are in the kernel of the composition $\Gamma \to \mathbb{Q} \otimes \theta \subset I(\theta) \to I(\theta)/\theta$. Note, however, that we are not free to conclude that the term $\frac{t^k-1}{m}V(B, Y)$ lies in this kernel.

Since this construction yields the well-defined element $[z, w]$, it must be independent of the choices made in the construction. This can also be verified purely algebraically, but since this is not needed here and in the interest of space, we defer these calculations to a forthcoming paper in which we will further develop the connection between Blanchfield and Farber–Levine pairings from a purely algebraic standpoint.

It is also a routine calculation from here to verify that the isometry class of $(T_{n-1}(\tilde{X}), [\,,\,])$ is determined algebraically completely by the isometry class of $(H_n(\tilde{C}), \langle\,,\,\rangle)$.

This completes the proof of Theorem 6.1. ∎

With a little more work, one could enlarge this theorem to apply to more general cases, for example some disk knots that are not necessarily simple. However, the theorem as stated will be sufficient for our purposes.

Now that we have shown that, for a simple disk knot, the Farber–Levine torsion pairing on $T_{n-1}(\tilde{X})$ is determined by the self-Blanchfield pairing on $H_n(\tilde{C})$, we wish to strengthen this result somewhat and show that, in fact, it only depends on the self-Blanchfield pairing on $\bar{H}$, the cokernel of the map $H_n(\tilde{X}) \to H_n(\tilde{C})$. This pairing will no longer determine all of $H_{n-1}(\tilde{X})$, but it suffices to determine $T_{n-1}(\tilde{X})$ and its Farber–Levine pairing. From this, we will be able to conclude that the Farber–Levine pairing is determined by the Seifert matrix of the disk knot.

Theorem 6.2: *For a simple disk knot $L : D^{2n-1} \hookrightarrow D^{2n+1}$, the Λ-module $T_{n-1}(\tilde{X})$ and its Farber–Levine $\mathbb{Z}$-torsion pairing are determined up to isometry by the isometry class of $\bar{H}$ with its self-Blanchfield pairing.*

Proof. Once again, we know that we have the exact sequence

$$0 \longrightarrow H_n(\tilde{X}) \xrightarrow{i_*} H_n(\tilde{C}) \xrightarrow{p_*} H_n(\tilde{C}, \tilde{X}) \longrightarrow H_{n-1}(\tilde{X}) \longrightarrow 0 \tag{2}$$

and that the modules $H_{n-1}(\tilde{X})$ and $T_{n-1}(\tilde{X})$ and the Farber-Levine pairing on $T_{n-1}(\tilde{X})$ are determined by the self-Blanchfield pairing on $H_n(\tilde{C})$. The module $\bar{H}$ is the cokernel of i_*, and it contains no $\mathbb{Z}$-torsion as $H_n(\tilde{C}, \tilde{X})$ is $\mathbb{Z}$-torsion free (since the knot is simple).

Consider now the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bar{H} & \xrightarrow{p} & \overline{\mathrm{Hom}_\Lambda(H_n(\tilde{C}), Q(\Lambda)/\Lambda)} & \xrightarrow{\partial} & H_{n-1}(\tilde{X}) & \longrightarrow & 0 \\
 & & {=}\big\uparrow & & \big\uparrow{\pi^*} & & \big\uparrow{g} & & \\
0 & \longrightarrow & \bar{H} & \xrightarrow{\rho} & \overline{\mathrm{Hom}_\Lambda(\bar{H}, Q(\Lambda)/\Lambda)} & \xrightarrow{\eta} & A & \longrightarrow & 0.
\end{array}$$

p denotes the map induced from p_*, and the first line is exact by the exactness of (2). Equation (2) also induces the map $\rho : \bar{H} \to \overline{\mathrm{Hom}_\Lambda(\bar{H}, Q(\Lambda)/\Lambda)}$ since the self-Blanchfield pairing is trivial on any element of $\mathrm{im}(i_*) = \ker(p_*)$. The map ρ is injective since the self-Blanchfield pairing on $\bar{H}$ is nondegenerate. The map π^* is induced by the projection $\pi : H_n(\tilde{C}) \to \bar{H}$, and it is injective since π is surjective and the Hom functor is left exact. The Λ-module A is the cokernel of ρ by definition, and g is induced by the rest of the diagram. g is injective by the five-lemma.

Suppose $x \in T_{n-1}(\tilde{X})$, the $\mathbb{Z}$-torsion submodule, and that $m = |T_{n-1}(\tilde{X})|$. From the diagram, $x = \partial(y)$ for some $y \in \overline{\mathrm{Hom}_\Lambda(H_n(\tilde{C}), Q(\Lambda)/\Lambda)}$, and $my = p(z)$ for some $z \in \bar{H}$. By commutativity, $my = \pi^*\rho(z)$. So my lifts to $\overline{\mathrm{Hom}_\Lambda(\bar{H}, Q(\Lambda)/\Lambda)}$, which means that my annihilates the subgroup $H_n(\tilde{X})$ of $H_n(\tilde{C})$. So for every element $w \in H_n(\tilde{X})$, $my(w) = V(w, my) \in \Lambda$, which implies that each rational function $y(w)$ must be of the form λ_w/m for some $\lambda_w \in \Lambda$.

We claim that in fact we must then have $y(w) \in \Lambda$ for every $w \in H_n(\tilde{X})$. The proof is similar to that of [15, Lemma 5.1]. Suppose that $w \in H_n(\tilde{X})$. Since $H_n(\tilde{X})$ is of type K, by the proof of [15, Corollary 1.3] there is a polynomial Δ such that $\Delta H_n(\tilde{X}) = 0$ and $\Delta(1) = \pm 1$. So $y(\Delta w) = \Delta y(w) = \Delta\lambda_w/m \in \Lambda$. But since $\Delta(1) = \pm 1$, no factor of m divides Δ in Λ, so it must be that m divides each λ_w, i.e. $y(w) = \lambda_w/m \in \Lambda$.

This shows that y annihilates $H_n(\tilde{X})$, which implies that y lifts to an element in $\overline{\mathrm{Hom}(\bar{H}, Q(\Lambda)/\Lambda)}$, i.e. $y = \pi^*(y')$, which implies that $x = \partial(y) = g\eta(y')$. So $T_{n-1}(\tilde{X}) \subset \mathrm{im}(g)$. But $T_{n-1}(\tilde{X})$ is finite [15, Lemma 3.1] and g is injective, so we must have $T_{n-1}(\tilde{X}) \cong T(A)$, the $\mathbb{Z}$-torsion subgroup of A. Thus $\bar{H}$ and its pairing, realized by ρ, determine $T_{n-1}(\tilde{X})$

The Farber–Levine $\mathbb{Z}$-torsion pairing on $T(A) \cong T_{n-1}(\tilde{X})$ is now determined by the self-Blanchfield pairing on $\bar{H}$ as in the proof of Theorem 6.1 and using the inclusion of the second row of the diagram into the first. So $T(A)$ and $T_{n-1}(\tilde{X})$ have isometric Farber–Levine pairings, induced by the self-Blanchfield pairing on $\bar{H}$. ∎

Corollary 6.3: *For a simple disk knot $L : D^{2n-1} \hookrightarrow D^{2n+1}$, the Λ-module $T_{n-1}(\tilde{X})$ and its Farber–Levine $\mathbb{Z}$-torsion pairing are determined up to isometry by any Seifert matrix for L.*

Proof. As seen in Section 5, the module $\bar{H}$ and its Blanchfield self-pairing are determined by any Seifert matrix for L. Thus the corollary follows immediately from the preceding theorem. ∎

7. Changing Seifert surfaces

The entirety of this section will be devoted to studying what happens to the Seifert matrix of a disk knot when we change the Seifert surface. Such alterations can always be performed by first doing surgery on the boundary Seifert surface F and then performing internal surgeries that avoid the boundary. Although we will see that different effects arise in different cases, we can summarize the results as follows:

Theorem 7.1: *Any two Seifert matrices for a disk knot differ by a rational S-equivalence.*

Proposition 3.11, which stated that two Seifert matrices for a disk knot are cobordant, follows.

To prove the theorem, we need to relate various Seifert surfaces for a fixed disk knot. So suppose that we have two copies of a disk knot L, which we will call L_1 and L_2, with Seifert surfaces V_1 and V_2 and boundary Seifert surfaces F_1 and F_2. Consider the knot $L \times I = (D^{2n+1} \times I, D^{2n-1} \times I)$. This is also a disk knot, and we can think of it as realizing the trivial cobordism from L_1 to L_2. On the boundary, $\partial D^{2n+1} \times I$, we have the trivial cobordism of the boundary knot K. As in [14, §3], we can then construct a cobordism U from F_1 to F_2 in $\partial D^N \times I$ such that ∂U is equal to the union of F_1, $-F_2$ and the trace of the trivial isotopy. The union $V_1 \cup U \cup -V_2$ is a Seifert surface for $L_1 \cup K \times I \cup -L_2$, the boundary knot of $L \times I$. By [13, §8], this can be extended to a Seifert surface W for $L \times I$. The pair (W, U) thus provides a cobordism from (V_1, F_1) to (V_2, F_2).

Now, as usual when dealing with cobordism with boundaries, we can break up the process into two distinct steps. We can first consider the cobordism of the boundary. In our case this amounts to beginning with $V_1 \subset D^{2n+1}$ and adjoining $U \subset \partial D^{2n+1} \times I$. In other words, we form $(D^{2n+1}, V_1) \cup_{(S^{2n}, F_1)} (S^{2n} \times I, U)$. Note that we do not need to mention the knots explicitly since they are contained in the embedding information. Then we perform the usual trick and "rekink" the diagram so that W becomes a cobordism rel boundary from $V_1 \cup U$ to V_2.

In the first subsection below, we consider the second stage and determine how a Seifert matrix is affected by an internal cobordism, i.e., one that leave the boundary Seifert surface fixed. In the second subsection, we consider the effect of the boundary cobordism.

7.1. Changing the Seifert surface on the interior. In this subsection, we first assume that we have two of the same disk knot $L : D^{2n-1} \hookrightarrow D^{2n+1}$ (denoted L_1 and L_2 when necessary) with two Seifert surfaces V_1 and V_2 that agree on the boundary (i.e., they have the same Seifert surface for the boundary sphere knot $F := F_1 = F_2$), then we can embed $L \times I$ in $D^{2n+1} \times I$ and consider the boundary knot $L_1 \cup -L_2 \cup K \times I$ and its Seifert surface $V_1 \cup -V_2 \cup F \times I$. This can be extended to a Seifert surface W for the whole disk knot $L \times I$, see [13, §8]. The analysis now of how the change in the Seifert matrix from that obtained from V_1 to that obtained from V_2 is highly analogous to the similar situation for sphere knots studied in [14]. Due to the similarity, we only sketch an outline of the proof, highlighting where generalizations occur.

One begins by separating W into critical levels using a smooth (PL) height function. This allows us to restrict to the case where W is obtained from V_1 by adding a single handle so that V_1 and V_2 differ by a single surgery, so we make this assumption.

Also as in [14], there is no effect to $F_n(V_1)$ if the index of the handle is less than n or if it has index n and the boundary of the cocore of the handle (which is ∂_* of a generator of $H_{n+1}(W, V_2)$) has finite order in $H_n(V_2)$. This follows via some basic surgery arguments and intersection number arguments.

If the handle is of index n and the boundary of the cocore, $a \in H_n(V_2)$, is not a torsion element and that a_0 is its primitive (i.e. a is a nontrivial positive multiple of a_0 and a_0 is not a multiple of any other element), it can be shown that either $\bar{E}$ and its pairing are unaffected or that $F_n(V_2) \cong F_n(V_1) \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $\bar{E} \cong \bar{E} \oplus \mathbb{Z}^2$, with the pairing restricted to $\bar{E}$ the original one. In fact, which case occurs depends only on whether or not $H_n(W, V_1; \mathbb{Q}) \to H_{n-1}(V_1, \mathbb{Q})$ is injective. This is also a consequence of basic surgery arguments, such as those in [14], and some extra diagram chasing that is necessary to account for V_1 and V_2 having nonspherical boundary.

The upshot is that we obtain a basis for $\bar{E}_2$ consisting of the basis $\{\delta_i\}_{i=1}^m$ for $\bar{E}_1$ (translated by homology up to V_2) plus a_0 and another new generator, b_0. Looking at the Seifert matrix for $\bar{E}_2$, Lemma 1 of [14] implies that the translated δ_i will have the same linking matrix θ_1 as they did for V_1. And since a_0 corresponds to the boundary of the cocore of the added handle, it is null-homologous in W and thus links trivially with all the translated δ_i and also with itself. Thus as in [14], we obtain a matrix for θ_2 of the following form:

$$\begin{pmatrix} \theta_1 & 0 & \eta \\ 0 & 0 & x \\ \xi & x' & y \end{pmatrix}, \tag{3}$$

where θ_1 is an $m \times m$ matrix, η is a $1 \times m$ matrix, ξ is an $m \times 1$ matrix, and x, x', and y are integers. The key difference from Levine's matrix [14, p. 188] is that here $x' + (-1)^n x$, which is the intersection number of a_0 and b_0, will not necessarily be ± 1. This is because V_2 does not necessarily have a spherical boundary, and so we do not have Poincaré duality to enforce the integral unimodularity of $\theta + (-1)^n \theta'$.

In fact, we can choose b_0 so that it maps to a multiple of the generator of $H_{n-1}(S^{n-1} \times S^n)$ under the boundary map of the Mayer–Vietoris sequence for the surgery representation of V_2. This implies that as a chain, b_0 can be represented by a multiple of the attached disk D^n suitably translated into V_2 plus another piece whose boundary is a multiple of the attaching S^{n-1}, also translated into V_2. Note that the intersection number of b_0 and a_0 is the smallest possible (in absolute value) nonzero intersection number between a_0 and all elements of $\bar{E}_2$: a_0 does not intersect any of the δ_i, since they all lie in V_0 and a_0 is the cocore of the handle. Nor does a_0 intersect itself, since the cocore can be pushed off itself along the handle. So no further changes of basis keeping a_0 fixed can provide a basis element that has a smaller nonzero intersection number with a_0 than b_0 does. Clearly, however, the intersection of a_0 and b_0 is nontrivial.

Now, from [7, §3.6], the Alexander polynomial $c_n(t)$ associated to the coimage of $H_n(\tilde{C};\mathbb{Q}) \to H_n(\tilde{C},\tilde{X};\mathbb{Q})$ and determined up to similarity in Λ is the determinant of $(-1)^{n+1}(R^{-1})'\tau R t - \tau' = (R^{-1})'((-1)^{n+1}\tau R t - R'\tau') = (R^{-1})'((-1)^{n+1}\theta' t - \theta)$. But recall that we also know that, with an appropriate integrally unimodular change of bases (which therefore will not affect its determinant), $-R = \theta + (-1)^n \theta'$, here R is just the transpose of the intersection matrix on $\bar{E}$. So the Alexander polynomial is the product of the determinants of

$$((-1)^{n+1}\theta' - \theta))^{-1} \quad \text{and} \quad (-1)^{n+1}\theta' t - \theta.$$

If we compare these polynomials as obtained using θ_2 and θ_1, we see that, just as in [14], the determinant of $((-1)^{n+1}\theta_2' t - \theta_2)$ is that of $((-1)^{n+1}\theta_1' t - \theta_1)$

multiplied by $((-1)^{n+1}xt - x')((-1)^{n+1}x't - x)$, and we also see that the determinant of $((-1)^{n+1}\theta_2' - \theta_2))^{-1}$ is that of $((-1)^{n+1}\theta_1' - \theta_1)^{-1}$ multiplied by $((-1)^{n+1}x - x')((-1)^{n+1}x' - x)$. Since this modification to the Seifert matrix cannot change the polynomial, which is an invariant of the knot, beyond multiplication by $\pm$ a power of t, it follows that either x' or x must be 0.

If it so happens that $x' + (-1)^n x = \pm 1$, then θ_2 and θ_1 are integrally S-equivalent as in [14]. In some cases, this will be guaranteed. For example, if the attaching sphere S^{n-1} is nullhomologous in V_2, then b_0 can be chosen so that the intersection of a_0 and b_0 is equal to 1. We already know that S^{n-1} cannot represent a free element of V_1, or else $\partial_* : \mathbb{Q} \cong H_n(W, V_1; \mathbb{Q}) \to H_{n-1}(V_1; \mathbb{Q})$ will be injective, which will imply that $H_n(W; \mathbb{Q}) \cong H_n(V_1; \mathbb{Q})$, which we know does not happen in the case under consideration. So the remaining case is that in which S^{n-1} is division null-homologous, but not null-homologous itself.

We know by Poincaré duality that there must be an element of $H_n(V_2, F)$ whose intersection with a_0 must be 1, and again this must be an element that is the sum of two chains, one of which is represented by the core of the handle (pushed into the boundary of the handle) and the other of which must have as boundary one piece that is the attaching sphere and another piece that is in F (this second piece cannot be empty, else S^{n-1} bounds in V_2, which is not true in the case under consideration). In other words, we see that in this case the attaching sphere must be homologous to a cycle in F. Thus this "bad" case, in which $x' + (-1)^n x \neq \pm 1$, can only happen if the attaching sphere represents a torsion element of $H_{n-1}(V_1)$ that is in the image of $H_{n-1}(F)$ under inclusion. In this case, we do not have S-equivalence, per se, but we do obtain a special type of elementary expansion of the form above, with either x or x' equal to 0 and the other equal to the intersection number of a_0 and b_0. We do obtain rational S-equivalence

This completes our study of what happens to the Seifert matrix when a handle of index $\leq n$ is added to the interior of V. But of course the addition of handles of higher index can be treated by reversing the direction of the cobordism. So this takes care of all surgeries on spheres in the interior of V.

7.2. Changing the boundary Seifert surface. It remains to consider those cobordisms that simply add to the boundary; we can consider this possibility in more detail. Again we can break the situation into the addition of one handle at a time by the usual Morse theory argument. So we must see the effect on the Seifert matrix of adding a handle to V along F. We will denote V plus this handle as V', we will let F' be the new resulting boundary piece after the surgery, and we will let F_0 represent F minus a neighborhood of the attaching sphere. Let $i : F \to V$, i_0, $i' : F' \to V'$, and $i_0 : F_0 \to V$ denote the inclusion maps.

We first prove that in most dimensions attaching a disk to V along F does not affect the Seifert matrix.

7.2.1. *Handles of index* $\neq \mathbf{n}, \mathbf{n+1}$. We consider attaching a handle of index $j \neq n, n+1$ so that $V' \sim_{h.e.} V \cup D^j$. Then $H_i(V', V) \neq 0$ if and only if $i = j$, so $H_i(V) \cong H_i(V')$ by inclusion for $i \neq j, j-1$. In particular, $H_n(V) \cong H_n(V')$ unless $j = n$ or $j = n+1$. Some elementary diagram chasing then reveals that $\operatorname{cok}(i : H_n(F) \to H_n(V)) \cong \operatorname{cok}(i' : H_n(F') \to H_n(V'))$ and that the Seifert matrices remain unchanged by the handle attachment.

This leaves the cases of $j = n$ and $j = n + 1$.

7.2.2. *Handles of index* $\mathbf{n}$. In this case $H_n(V', V) \cong \mathbb{Z}$ and $H_i(V', V) = 0$ otherwise. This implies that $H_n(V) \to H_n(V')$ is injective, and either it is an isomorphism or the inclusion of a direct summand, the other summand being $\mathbb{Z}$.

Case: $H_n(V) \cong H_n(V')$. Assume that $H_n(V) \cong H_n(V')$. This will be the case if $\partial_* : H_n(V', V) \to H_{n-1}(V)$ is injective, which will happen if the attaching sphere for the handle generates a free subgroup of $H_{n-1}(V)$.

The Mayer-Vietoris sequences for F and F' become

(4)

$$0 \longrightarrow H_n(F_0) \longrightarrow H_n(F) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\Phi} H_{n-1}(F_0) \oplus \mathbb{Z} \longrightarrow H_{n-1}(F) \longrightarrow 0$$

$$0 \longrightarrow H_n(F_0) \longrightarrow H_n(F') \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_{n-1}(F_0) \oplus \mathbb{Z} \longrightarrow H_{n-1}(F') \longrightarrow 0.$$

By chasing through these sequences and the long exact sequence of (F', F_0), one establishes that $\operatorname{cok}(i_0) \cong \operatorname{cok}(i')$.

We next consider the exact sequence of the pair (F, F_0). By excision, $H_i(F, F_0) \cong H_i(S^{n-1} \times D^n, S^{n-1} \times S^{n-1})$. So again $H_n(F_0) \to H_n(F)$ is injective, and $H_n(F, F_0) \cong \mathbb{Z}$ is generated by the cell $* \times D^n$.

Subcase: $H_n(F_0) \twoheadrightarrow H_n(F)$. $H_n(F_0) \to H_n(F)$ will be surjective if the boundary of the cell $* \times D^n$, the boundary of a fiber of the normal disk bundle of the attaching sphere, generates an infinite cyclic group in $H_{n-1}(F_0)$. In this case, one easily verifies $\mathrm{cok}(i_0) \cong \mathrm{cok}(i)$ Thus together with the previous calculation that $\mathrm{cok}(i_0) \cong \mathrm{cok}(i')$, we have $\mathrm{cok}(i) \cong \mathrm{cok}(i')$, and since all of these vertical maps have been by inclusions, each cokernel can employ the same chains as generators, whence the Seifert matrices are identical.

Subcase: Not $H_n(F_0) \twoheadrightarrow H_n(F)$. In the alternative case in which a multiple of this fiber sphere bounds in F_0, there is a splitting and $H_n(F) \cong H_n(F_0) \oplus \mathbb{Z}$. The $\mathbb{Z}$ term can be generated by the sum of two chains, one lying in F_0 and one in $S^{n-1} \times D^n$, both of whose boundary chains are corresponding (opposite sign) multiples of the fiber sphere (of course the one not in F_0 will just be a multiple of the fiber disk). This can also be seen from the Mayer–Vietoris sequence. Call this generator a. If a bounds in V, then $H_n(F_0) \to H_n(F)$ will be onto the coimage of $H_n(F) \to H_n(V)$ and it will follow again that $\mathrm{cok}(i) \cong \mathrm{cok}(i')$. Similarly, if the image of a in $H_n(V)$ is torsion, then $H_n(F_0) \to H_n(F)$ will be onto the coimage of $H_n(F) \to H_n(V)$ mod torsion. Again we get $\mathrm{cok}(i) \cong \mathrm{cok}(i')$ and isometric pairings.

So the one remaining case of interest in this subcase will be that in which the image of a generates an infinite cyclic group in V. Note that, since $H_n(F_0) \to H_n(F)$ is injective, $H_n(F_0) \to H_n(V)$ actually factors through $H_n(F)$ so that the image of the $H_n(F_0)$ summand of $H_n(F)$ will agree with the image of $H_n(F_0)$.

We will actually see that a multiple of the image of a in $H_n(V)$ lies in the image of $H_n(F_0)$. This will imply that $\mathrm{cok}(i) \cong \mathrm{cok}(i_0)$ mod torsion, and it will follow that the Seifert matrix is unchanged by the addition of the handle. To prove the claim, we consider the image of a in $H_n(V)$, still represented by the chain a as described above. Since the inclusion $H_n(V) \cong H_n(V')$ is an isomorphism, a must represent an infinite cyclic subgroup of $H_n(V')$. The image of this homology class in $H_n(V', F')$, also represented by (the appropriate coset of) a, must be 0 for the following reason. By duality, we know that $H_n(V')$ and $H_n(V', F')$ are dually paired by the intersection form. But our chain representing a in $H_n(V', F')$ can be made disjoint from any other chain representing a class in $H_n(V')$ since all such classes can be assumed to lie in V and hence the interior of V using the inclusion-induced isomorphism $H_n(V) \cong H_n(V')$ and by pushing in along a collar of the boundary F of V. But a lies in F and hence is disjoint from any such chain. We conclude that

a represents a torsion element in $H_n(V', F')$. Thus some multiple of a must be in the image of $H_n(F') \to H_n(V')$, and hence the image of the composite $H_n(F_0) \xrightarrow{\cong} H_n(F') \to H_n(V')$. So some multiple of a is representable by a chain lying entirely in F_0. By these geometric arguments, or by chasing the diagram around algebraically, we see that some multiple of $a \in H_n(V)$ is in the image of $H_n(F_0)$. So a goes to a torsion element in $\text{cok}(i)$ and so 0 in $\text{cok}(i)$ mod torsion.

Case: $H_n(V) \not\cong H_n(V')$. We next consider the case in which $H_n(V) \not\cong H_n(V')$. This happens if $\mathbb{Z} \cong H_n(V', V) \to H_{n-1}(V)$ has nontrivial kernel, i.e. if a multiple of the attaching sphere bounds in V. In this case, $H_n(V') \cong H_n(V) \oplus \mathbb{Z}$, the additional $\mathbb{Z}$ summand can be taken as generated by a chain C consisting of a multiple of the core of the attached disk D^n and a chain in V whose boundary is a multiple of the attaching sphere. By pushing in along a collar of ∂V, we can assume that the geometric intersection of this chain C with F is the attaching sphere. C is well-defined in this way up to a cycle in V, but we can fix a specific one as a generator of the summand.

Subcase: $H_n(F') \cong H_n(F_0) \oplus \mathbb{Z}$. Suppose that the translate of the attaching sphere, $\partial(D^n \times *)$, $* \in S^{n-1}$, weakly bounds in F_0. Then from the long exact sequence of the pair (F', F_0), we see that $H_n(F') \cong H_n(F_0) \oplus \mathbb{Z}$. This follows since $H_n(F', F_0) \cong H_n(D^n \times S^{n-1}, S^{n-1} \times S^{n-1}) \cong \mathbb{Z}$, using excision and the long exact sequence of the latter pair. The distinguished $\mathbb{Z}$ summand of $H_n(F') \cong H_n(F) \oplus \mathbb{Z}$ can then be generated by a chain B composed of a multiple of a translate of the core of the handle and another chain in F_0 whose boundary coincides with that of this multiple of the core. B is well-defined up to cycles in F_0, and again we fix a representative. The image in $H_n(V', V)$ of the chain B represents a non–trivial multiple of the generator.

We will study $\text{cok}(i')$ and $\text{cok}(i_0)$ modulo torsion. Writing $H_n(F') \cong H_n(F_0) \oplus \mathbb{Z}$ and $H_n(V') \cong H_n(V) \oplus \mathbb{Z}$, we have clearly that $i'(x, 0) = (i_0(x), 0)$, since the image of F_0 is in V and hence all such elements go to 0 under the surjection $H_n(V') \to \mathbb{Z} \cong H_n(V', V)$. We also have that $i'(0, B) = (y, z)$, where y is unknown at this point, but z must be nonzero, since, again, we know that B represents a nontrivial multiple of the generator of $H_n(V', V)$. It follows by an application of the serpent lemma that $\text{cok}_{\mathbb{Q}}(i_0) \cong \text{cok}_{\mathbb{Q}}(i')$ and that, with $\mathbb{Q}$ coefficients, $\text{cok}_{\mathbb{Q}}(i_0) \cong \text{cok}_{\mathbb{Q}}(i')$.

Now let us look at $H_n(F_0;\mathbb{Q}) \to H_n(F;\mathbb{Q})$. This is also an injection by the long exact sequence of the pair. Suppose it is not an isomorphism. Then from the long exact sequence of the pair, $H_n(F;\mathbb{Q}) \cong H_n(F_0;\mathbb{Q}) \oplus \mathbb{Q}$. A generator A of the distinguished $\mathbb{Q}$ can be represented by a chain contained in F consisting of a multiple of a fiber of the tubular neighborhood of the attaching disk plus a chain in F_0 with the opposite boundary. This is because the existence of this extra term implies that a multiple of the boundary of the fiber bounds in F_0. We will see that this situation actually cannot arise.

In $H_n(V', F')$, the image of A is clearly homologous to a multiple of the relative cycle generated by the cocore of the handle, and, by the assumptions of this case leading to the nontriviality and nontorsion of C, the intersection of A and C cannot be 0, and it would follow that this image of A generates an infinite cyclic subgroup of $H_n(V', F';\mathbb{Q})$. So under the maps $H_n(F) \to H_n(V) \to H_n(V') \to H_n(V', F')$, A must map to a nontrivial element. Thus A maps to some element $0 \neq x \in H_n(V;\mathbb{Q})$, which maps to $0 \neq (x, 0) \in H_n(V';\mathbb{Q})$. Now consider the image of x in $H_n(V', F')$. This element is still represented by A, modulo chains in F'. The intersection of A with any cycle in V is 0, since any such cycle can be pushed into the interior of V and thus be made disjoint from F and F'. Now consider the intersection of A with C. We know that $i'(0, B) = (y, z)$, where $z = mC$ for some $m \in \mathbb{Q}$. But then the intersection of A with (y, z) is 0, since (y, z) goes to 0 in $H_n(V', F')$ and since A is the image of an element of $H_n(V')$. But this implies that the intersection of A with y is the negative of its intersection with z. But the intersection of A with y is 0 since y is in $H_n(V)$. Thus the intersection of A with z is 0, and so the intersection of A with C is 0. It then follows that A must map to 0 in $H_n(V', F';\mathbb{Q})$ since $H_n(V', F';\mathbb{Q})$ and $H_n(V';\mathbb{Q})$ are dual under the intersection pairing. So we arrive at a contradiction. Thus it must be in fact that $H_n(F) \cong H_n(F_0)$.

So we see that that $\operatorname{cok}(i) \cong \operatorname{cok}(i_0)$. However, we still have that $\operatorname{cok}(i_0) \to \operatorname{cok}(i')$ may only be an injection, the cokernel of **this** map being a cyclic torsion group. We can assume by changing basis if necessary that, modulo torsion, this map is represented by a matrix that is 0 except on the diagonal, all diagonal entries except perhaps the last one being equal to 1. The last entry is nonzero, say p, but may not be 1. So now all other basis elements of $\operatorname{cok}(i')$ but the last are represented by the chains that represent them in $\operatorname{cok}(i)$ mod torsion and so their linking pairings with each other remain unchanged.

The last basis element is homologous to $1/p$ times a chain lying in $\mathrm{cok}(i)$. So each of its linking numbers will simply be $1/p$ times those for the corresponding chain in $\mathrm{cok}(i)$. Hence the change to the Seifert matrix is to multiply the last row and column by $1/p$. In other words, the Seifert matrix changes by a rational change of bases, although the new matrix must also be integral.

Subcase: $H_n(F_0) \cong H_n(F') \cong H_n(F)$.

Suppose $H_n(F_0) \cong H_n(F')$. In this case, we show first that it is impossible to also have $H_n(F_0) \cong H_n(F)$, induced by inclusion. So suppose that $H_n(F_0) \cong H_n(F') \cong H_n(F)$, both isomorphisms induced by inclusion of F_0. Then the attaching sphere must generate a torsion (or zero) subgroup of $H_{n-1}(F)$. This is because all cycles of $H_n(F)$ can be homotoped into the interior of F_0 so that the intersection of the attaching sphere with any such cycle is empty. Thus, by the Poincaré duality of the $2n-1$ manifold ∂V, whose homology in all but the top dimension is equal to the homology of F, the attaching sphere cannot generate a free subgroup of $H_{n-1}(F)$. It follows that some multiple of the attaching sphere must bound in F. Thus, in rational homology, in which $H_n(V';\mathbb{Q}) \cong H_n(V;\mathbb{Q}) \oplus \mathbb{Q}$, the distinguished $\mathbb{Q}$ summand can be taken as generated by a cycle C composed of the attaching disk and a chain in F whose boundary is the (negative of) the attaching sphere. A multiple of C will generate the corresponding distinguished $\mathbb{Z}$ term with $\mathbb{Z}$ coefficients.

Okay, so now if $H_n(F_0) \cong H_n(F)$, $\mathrm{cok}(i) \cong \mathrm{cok}(i_0)$, integrally or rationally and generated by the same cycles in F_0. And since $H_n(F_0) \cong H_n(F')$, also generated by the same cycles, $\mathrm{im}(i_0) = \mathrm{im}(i') \subset H_n(V) \subset H_n(V')$, so we see that $\mathrm{cok}_{\mathbb{Q}}(i') \cong \mathrm{cok}_{\mathbb{Q}}(i_0) \oplus \mathbb{Q}$, the distinguished $\mathbb{Q}$ summand again generated by C. So the rational Seifert matrix for V' has one more row and column than that for V, and except for this row and column is identical to that for V. In this row and column, all except possibly the diagonal entry must be 0 because C cannot link any element in V. This is because in the process of putting a cobordism on F, we have extended the knot originally in D^{2n+1} to be in $D^{2n+1} \cup S^{2n} \times I$. The cobordism from F lies in $S^{2n} \times I$, and hence so does C. But all element representing cycles from $H_n(V)$ lie in the original D^{2n+1}. Since the n-dimensional homology groups of both D^{2n+1} and $S^{2n} \times I$ are trivial, cycles in each can bound entirely within each (and we can push along some collars if necessary). So C need not link anything from $H_n(V)$. Thus the rational Seifert matrix is 0 along the additional row and column except where they meet.

But now this must violate the invariance of the Alexander polynomial, which can be computed from the rational Seifert matrix. If the diagonal term is 0 or if n is odd, then $R = -\theta' + (-1)^{n+1}\theta$ is singular, which is impossible. If the diagonal term is not 0, say it is $x \neq 0$, then the Alexander polynomial will be altered by multiplication by $\frac{xt+x}{2x} = \frac{t+1}{2}$, which is also impossible as this term is not a rational multiple of a power of t and hence not a unit in the ring of rational Laurent polynomials.

Subcase: $H_n(F') \cong H_n(F_0)$ but $H_n(F_0) \not\cong H_n(F)$.

In this case, $H_n(F) \cong H_n(F_0) \oplus \mathbb{Z}$, from the long exact sequence of (F, F_0). The $\mathbb{Z}$ term can be taken as generated by a chain A that is the sum of a multiple of the fiber disk of the tubular neighborhood of the attaching sphere and another chain in F_0 with the opposite boundary.

The chain A must generate an infinite cyclic summand in $H_n(V)$ because, under the composition $H_n(F) \to H_n(V) \to H_n(V') \to H_n(V', F')$, A becomes relatively homologous to a multiple of the cocore of the attached handle, and this cocore must have a non-zero intersection number with any chain generating the distinguished $\mathbb{Z}$ summand of $H_n(V') \cong H_n(V) \oplus \mathbb{Z}$. We do not run here into the contradiction of the previous similar case since it is no longer true that a multiple of the generator of this summand of $H_n(V')$ is in the image of i', since now the image of i' in $H_n(V')$ must equal the image of i_0 in $H_n(V) \subset H_n(V')$. Meanwhile, the image of A in $H_n(V)$ must not be in the image of $H_n(F_0)$, since the composition $H_n(F_0) \cong H_n(F') \to H_n(V') \to H_n(V', F')$ is 0, and we know that the image of $H_n(F_0)$ in $H_n(V')$ is the same as the image of $H_n(F_0)$ in $H_n(V) \subset H_n(V')$. So we see that, in fact, A generates an infinite cyclic group in $H_n(V)$ that is not in the image of $H_n(F_0)$. So, mod torsion, $\text{cok}(i_0) \cong \text{cok}(i) \oplus \mathbb{Z}$.

It also follows from the serpent lemma that $\text{cok}(i') \cong \text{cok}(i_0) \oplus \mathbb{Z} \cong \text{cok}(i) \oplus \mathbb{Z}^2$.

Thus we see that the Seifert matrix for V' has two more rows and columns than the one for V, and, excluding these rows and columns, the matrices agree. We must now determine what entries go in these last two rows and columns for V'. By changing bases if necessary, we can assume that A is a multiple of a generator of the distinguished $\mathbb{Z}$ term of $\text{cok}(i_0) \cong \text{cok}(i) \oplus \mathbb{Z}$. But as in the previous case, we see that A, because it lies in F, does not link with any of the cycles in $H_n(V)$ including itself. It can only possibly link nontrivially with a chain generating the distinguished $\mathbb{Z}$ summand of $H_n(V') \cong H_n(V) \oplus \mathbb{Z}$. The same is then true for the generator of the summand containing A. Thus the matrix for V' must differ from that for V as in equation (3). The same arguments then show that we must have a rational S-equivalence.

7.2.3. *Handles of index* $\mathbf{n+1}$. Consider again the long exact sequence for (F, F_0). By excision, $H_i(F, F_0) \cong H_i(S^n \times D^{n-1}, S^n \times S^{n-2})$. Clearly, $H_{n+1}(S^n \times D^{n-1}) = H_{n-1}(S^n \times S^{n-2}) = 0$, and furthermore, $H_n(S^n \times S^{n-2}) \cong H_n(S^n \times D^{n-1}) \cong \mathbb{Z}$, the isomorphism being induced by inclusion and taking a generator $S^n \times * \subset S^n \times S^{n-2}$ to a generator $S^n \times * \subset S^n \times D^{n-1}$. It follows that $H_i(S^n \times D^{n-1}, S^n \times S^{n-2})$ and hence $H_i(F, F_0)$ is 0 for $i = n, n+1$. Thus $H_n(F_0) \cong H_n(F)$, induced by inclusion. It follows that $\text{cok}(i) \cong \text{cok}(i_0)$.3istu

On the other hand, we consider the Mayer–Vietoris sequence for F' and F_0. Since $H_{n-1}(S^n \times S^{n-2}) = H_n(D^{n+1} \times S^{n-2}) = 0$, the inclusion-induced homomorphism $H_n(F_0) \to H_n(F)$ is onto, possibly with kernel represented by the attaching sphere, appropriately translated to $S^n \times * \subset S^n \times S^{n-2} \subset F_0$.

Meanwhile, since V' is obtained from V by attaching an $n+1$ handle, $H_i(V, V')$ is 0 for $i \neq n+1$ and $\mathbb{Z}$ for $i = n+1$. Thus $H_n(V) \to H_n(V')$ is also onto, and its kernel is also generated by the attaching sphere. If the class of the attaching sphere is either trivial or torsion in $H_n(V)$, then $H_n(V) \to H_n(V')$ is an isomorphism mod torsion, and we obtain a diagram

$$\begin{array}{ccc} H_n(F_0) & \xrightarrow{i_0} & F_n(V) \\ {\scriptstyle\text{onto}}\downarrow & & \downarrow{\scriptstyle\cong} \\ H_n(F') & \xrightarrow{i'} & F_n(V'). \end{array}$$

Again we see that $\text{cok}(i_0) \cong \text{cok}(i)$, and again, since all maps are induced by inclusions, the Seifert pairing is unchanged.

If the attaching sphere generates an infinite cyclic subgroup of $H_n(V)$, it must also generate an infinite cyclic subgroup of $H_n(F_0)$ (if some multiple of it bounds in F_0, then that multiple also bounds in V since $F_0 \subset V$). So we have the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_n(F_0) & \longrightarrow & H_n(F') & \longrightarrow & 0 \\ & & {\scriptstyle\cong}\downarrow & & {\scriptstyle i_0}\downarrow & & {\scriptstyle i'}\downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_n(V) & \longrightarrow & H_n(V') & \longrightarrow & 0, \end{array}$$

in which both $\mathbb{Z}$ summands are generated by the attaching sphere. It follows now from the serpent lemma that $\mathrm{cok}(i_0) \cong \mathrm{cok}(i')$. It once more follows that the Seifert matrix is unchanged.

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